

Approximate Capacity of a Class of Partially Connected Interference Channels

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Abstract—We derive inner and outer bounds on the capacity region for a class of three-user partially connected interference channels. We focus on the impact of topology, interference alignment, and interplay between interference and noise. The representative channels we consider are the ones that have clear interference alignment gain. For these channels, Z-channel type outer bounds are tight to within a constant gap from capacity. We present near-optimal achievable schemes based on rate-splitting and lattice alignment.

Index Terms—Interference channel, interference alignment, nested lattice code, side information graph, topological interference management.

I. INTRODUCTION

A. Motivation

The capacity of the Interference channel remains one of the most challenging open problems in the domain of network information theory. The capacity region is not known in general, except for a specific range of channel parameters. For the two-user scalar Gaussian interference channel, where the interference alignment is not required, the approximate capacity region to within one bit is known [1]. For the channels where interference alignment is required such as the K -user Gaussian interference channel [2]–[5], [7], [11] and the Gaussian X-channel [9]–[11], a tight characterization of the capacity region is not known, even for symmetric channel cases.

A tractable approach to the capacity of interference channels is to consider partial connectivity of interference links and analyze the impact of topology on the capacity. Topological interference management [8] approach gives important insights on the degrees-of-freedom (DoF) of partially connected interference channels and their connection to index coding problems [18]–[25]. It is shown that the symmetric DoF of a partially connected interference channel can be found by solving the corresponding index coding problem.

In this paper, we consider a class of three-user partially connected interference channels and characterize approximate capacity regions at finite SNR. We focus on the impact of interference topology, interference alignment, and interplay between interference and noise. We choose a few representative topologies where we can achieve clear interference alignment gain. For these topologies, Z-channel type outer bounds are tight to within a constant gap from the corresponding inner bound. For each topology, we present an achievable scheme

based on rate-splitting, lattice alignment, and successive decoding.

B. Related Work

Lattice coding based on nested lattices is shown to achieve the capacity of the single user Gaussian channel in [12], [27]. The idea of lattice-based interference alignment by decoding the sum of lattice codewords appeared in the conference version of [4]. This lattice alignment technique is used to derive capacity bounds for three-user interference channel in [2], [3]. The idea of decoding the sum of lattice codewords is also used in [13]–[15] to derive the approximate capacity of the two-way relay channel. An extended approach, compute-and-forward [16], [17] enables to first decode some linear combinations of lattice codewords and then solve the lattice equation to recover the desired messages. This approach is also used in [7] to characterize approximate sum-rate capacity of the fully connected K -user interference channel.

The idea of sending multiple copies of the same sub-message at different signal levels, so-called Zigzag decoding, appeared in [5] where receivers collect side information and use them for interference cancellation.

The K -user cyclic Gaussian interference channel is considered in [6] where an approximate capacity for the weak interference regime ($\text{SNR}_k \geq \text{INR}_k$ for all k) and the exact capacity for the strong interference regime ($\text{SNR}_k \leq \text{INR}_k$ for all k) are derived. Our type 4 and 5 channels are $K = 3$ cases in *mixed* interference regimes, which were not considered in [6].

C. Main Results

We consider five channel types defined in Table I and described in Fig. 1 (a)–(e). Each channel type is a partially connected three-user Gaussian interference channel. Each transmitter is subject to power constraint $\mathbb{E}[X_k^2] \leq P_k = P$. Let us denote the noise variance by $N_k = \mathbb{E}[Z_k^2]$. Without loss of generality, we assume that $N_1 \leq N_2 \leq N_3$.

Definition 1 (side information graph): The side information graph representation of an interference channel satisfies the following.

- A node represents a transmitter-receiver pair, or equivalently, the message.
- There is a directed edge from node i to node j if transmitter i does not interfere at receiver j .

The side information graphs for five channel types are described in Fig. 1 (f)–(j). We state the main results in the

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Type	Channel model
1	$Y_1 = X_1 + X_2 + Z_1$ $Y_2 = X_1 + X_2 + X_3 + Z_2$ $Y_3 = X_2 + X_3 + Z_3$
2	$Y_1 = X_1 + X_2 + X_3 + Z_1$ $Y_2 = X_1 + X_2 + Z_2$ $Y_3 = X_1 + X_3 + Z_3$
3	$Y_1 = X_1 + X_3 + Z_1$ $Y_2 = X_2 + X_3 + Z_2$ $Y_3 = X_1 + X_2 + X_3 + Z_3$
4	$Y_1 = X_1 + X_3 + Z_1$ $Y_2 = X_1 + X_2 + Z_2$ $Y_3 = X_2 + X_3 + Z_3$
5	$Y_1 = X_1 + X_2 + Z_1$ $Y_2 = X_2 + X_3 + Z_2$ $Y_3 = X_1 + X_3 + Z_3$

TABLE I
FIVE CHANNEL TYPES

following two theorems, of which the proofs will be given in the main body of the paper.

Theorem 1 (Capacity region outer bound): For the five channel types, if (R_1, R_2, R_3) is achievable, it must satisfy

$$\sum_{j \in \mathcal{K}} R_j \leq \frac{1}{2} \log \left(1 + \frac{|\mathcal{K}|P}{\min_{j \in \mathcal{K}} \{N_j\}} \right) \quad (1)$$

for every subset \mathcal{K} of the nodes $\{1, 2, 3\}$ that does not include a directed cycle in the side information graph over the subset.

Theorem 2 (Capacity region to within one bit):

For any rate triple (R_1, R_2, R_3) on the boundary of the outer bound region, the point $(R_1 - 1, R_2 - 1, R_3 - 1)$ is achievable.

D. Paper Organization and Notation

The capacity outer bounds are derived in Section II. The inner bounds for each channel type and the corresponding gap analysis are given in Section III, IV, V, VI, VII, respectively. Section VIII concludes the paper. While lattice coding-based achievable rate regions for channel types 4 and 5 are presented in Section VI and VII, random coding achievability is given in Appendix.

Signal \mathbf{x}_{ij} is a coded version of message M_{ij} with code rate R_{ij} unless otherwise stated. The single user capacity at receiver k is denoted by $C_k = \frac{1}{2} \log \left(1 + \frac{P}{N_k} \right)$. Let \mathcal{C} denote the capacity region of an interference channel. Also, let \mathcal{R}_i and \mathcal{R}_o denote the capacity inner bound and the capacity outer bound, respectively. Thus, $\mathcal{R}_i \subset \mathcal{C} \subset \mathcal{R}_o$. Let δ_k denote the gap on the rate R_k between \mathcal{R}_i and \mathcal{R}_o . Let δ_{jk} denote the gap on the sum-rate $R_j + R_k$ between \mathcal{R}_i and \mathcal{R}_o . For example, if

$$\mathcal{R}_i = \{(R_j, R_k) : R_k \leq L_k, R_j + R_k \leq L_{jk}\} \quad (2)$$

$$\mathcal{R}_o = \{(R_j, R_k) : R_k \leq U_k, R_j + R_k \leq U_{jk}\}, \quad (3)$$

then $\delta_k = U_k - L_k$ and $\delta_{jk} = U_{jk} - L_{jk}$. For side information graph, we use graph notation of [23]. For example, $\mathcal{G}_1 = \{(1|3), (2), (3|1)\}$ means that node 1 has an incoming edge from node 3, that node 2 has no incoming edge, and that node 3 has an incoming edge from node 1.

II. CAPACITY OUTER BOUNDS

We prove the capacity outer bound in *Theorem 1* for each channel type. The result is summarized in Table II. The shape of the outer bound region is illustrated in Fig. 2. For all channel types, we assume $P_1 = P_2 = P_3 = P$ and $N_1 \leq N_2 \leq N_3$.

A. Channel Type 1

In this section, we present an outer bound on the capacity region of Type 1 channel defined by

$$\begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} + \begin{bmatrix} Z_1 \\ Z_2 \\ Z_3 \end{bmatrix}.$$

We state the outer bound in the following theorem.

Theorem 3: The capacity region of Type 1 channel is contained in the following outer bound region:

$$\begin{aligned} R_k &\leq C_k, \quad k = 1, 2, 3 \\ R_1 + R_2 &\leq \frac{1}{2} \log \left(1 + \frac{P}{N_1} \right) + \frac{1}{2} \log \left(\frac{2P + N_2}{P + N_2} \right) \\ R_2 + R_3 &\leq \frac{1}{2} \log \left(1 + \frac{P}{N_2} \right) + \frac{1}{2} \log \left(\frac{2P + N_3}{P + N_3} \right). \end{aligned}$$

Proof: The individual rate bounds are obvious. We proceed to sum-rate bounds.

$$\begin{aligned} n(R_1 + R_2 - \epsilon) &\leq I(X_1^n; Y_1^n) + I(X_2^n; Y_2^n) \\ &\leq I(X_1^n; Y_1^n | X_2^n) + I(X_2^n; Y_2^n | X_3^n) \\ &= h(Y_1^n | X_2^n) - h(Y_1^n | X_1^n, X_2^n) \\ &\quad + h(Y_2^n | X_3^n) - h(Y_2^n | X_2^n, X_3^n) \\ &= h(X_1^n + Z_1^n) - h(Z_1^n) \\ &\quad + h(X_1^n + X_2^n + Z_2^n) - h(X_1^n + Z_2^n) \\ &\leq \frac{n}{2} \log \left(\frac{P + N_1}{N_1} \right) + \frac{n}{2} \log \left(\frac{2P + N_2}{P + N_2} \right) \end{aligned}$$

where the first inequality is by Fano's inequality, the second inequality due to the independence of X_1, X_2, X_3 . The third inequality holds from the fact that Gaussian distribution maximizes differential entropy and that $h(X_1^n + Z_1^n) - h(X_1^n + Z_2^n)$ is also maximized by Gaussian distribution. Similarly,

$$\begin{aligned} n(R_2 + R_3 - \epsilon) &\leq I(X_2^n; Y_2^n) + I(X_3^n; Y_3^n) \\ &\leq I(X_2^n; Y_2^n | X_1^n, X_3^n) + I(X_3^n; Y_3^n) \\ &= h(Y_2^n | X_1^n, X_3^n) - h(Y_2^n | X_1^n, X_2^n, X_3^n) \\ &\quad + h(Y_3^n) - h(Y_3^n | X_3^n) \\ &= h(X_2^n + Z_2^n) - h(Z_2^n) \\ &\quad + h(X_2^n + X_3^n + Z_3^n) - h(X_2^n + Z_3^n) \\ &\leq \frac{n}{2} \log \left(\frac{P + N_2}{N_2} \right) + \frac{n}{2} \log \left(\frac{2P + N_3}{P + N_3} \right). \end{aligned}$$

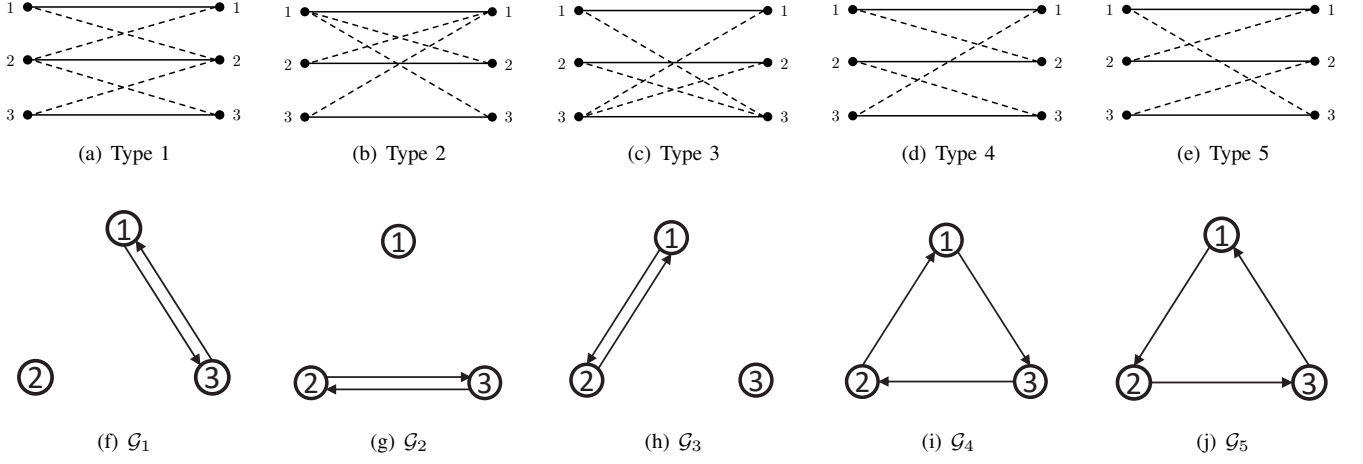


Fig. 1. Five channel types and their side information graphs: $\mathcal{G}_1 = \{(1|3), (2), (3|1)\}$, $\mathcal{G}_2 = \{(1), (2|3), (3|2)\}$, $\mathcal{G}_3 = \{(1|2), (2|1), (3)\}$, $\mathcal{G}_4 = \{(1|2), (2|3), (3|1)\}$, and $\mathcal{G}_5 = \{(1|3), (2|1), (3|2)\}$.

B. Channel Type 2

In this section, we present an outer bound on the capacity region of Type 2 channel defined by

$$\begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} + \begin{bmatrix} Z_1 \\ Z_2 \\ Z_3 \end{bmatrix}.$$

We state the outer bound in the following theorem.

Theorem 4: The capacity region of Type 2 channel is contained in the following outer bound region:

$$\begin{aligned} R_k &\leq C_k, \quad k = 1, 2, 3 \\ R_1 + R_2 &\leq \frac{1}{2} \log \left(1 + \frac{P}{N_1} \right) + \frac{1}{2} \log \left(\frac{2P + N_2}{P + N_2} \right) \\ R_1 + R_3 &\leq \frac{1}{2} \log \left(1 + \frac{P}{N_1} \right) + \frac{1}{2} \log \left(\frac{2P + N_3}{P + N_3} \right). \end{aligned}$$

Proof:

$$\begin{aligned} n(R_1 + R_2 - \epsilon) &\leq I(X_1^n; Y_1^n) + I(X_2^n; Y_2^n) \\ &\leq I(X_1^n; Y_1^n | X_2^n, X_3^n) + I(X_2^n; Y_2^n) \\ &= h(Y_1^n | X_2^n, X_3^n) - h(Y_1^n | X_1^n, X_2^n, X_3^n) \\ &\quad + h(Y_2^n) - h(Y_2^n | X_2^n) \\ &= h(X_1^n + Z_1^n) - h(Z_1^n) \\ &\quad + h(X_1^n + X_2^n + Z_2^n) - h(X_1^n + Z_2^n) \\ &\leq \frac{n}{2} \log \left(\frac{P + N_1}{N_1} \right) + \frac{n}{2} \log \left(\frac{2P + N_2}{P + N_2} \right). \end{aligned}$$

$$\begin{aligned} n(R_1 + R_3 - \epsilon) &\leq I(X_1^n; Y_1^n) + I(X_3^n; Y_3^n) \\ &\leq I(X_1^n; Y_1^n | X_2^n, X_3^n) + I(X_3^n; Y_3^n) \\ &= h(Y_1^n | X_2^n, X_3^n) - h(Y_1^n | X_1^n, X_2^n, X_3^n) \\ &\quad + h(Y_3^n) - h(Y_3^n | X_3^n) \\ &= h(X_1^n + Z_1^n) - h(Z_1^n) \\ &\quad + h(X_1^n + X_3^n + Z_3^n) - h(X_1^n + Z_3^n) \\ &\leq \frac{n}{2} \log \left(\frac{P + N_1}{N_1} \right) + \frac{n}{2} \log \left(\frac{2P + N_3}{P + N_3} \right). \end{aligned}$$

C. Channel Type 3

In this section, we present an outer bound on the capacity region of Type 3 channel defined by

$$\begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} + \begin{bmatrix} Z_1 \\ Z_2 \\ Z_3 \end{bmatrix}.$$

We state the outer bound in the following theorem.

Theorem 5: The capacity region of Type 3 channel is contained in the following outer bound region:

$$\begin{aligned} R_k &\leq C_k, \quad k = 1, 2, 3 \\ R_1 + R_3 &\leq \frac{1}{2} \log \left(1 + \frac{P}{N_1} \right) + \frac{1}{2} \log \left(\frac{2P + N_3}{P + N_3} \right) \\ R_2 + R_3 &\leq \frac{1}{2} \log \left(1 + \frac{P}{N_2} \right) + \frac{1}{2} \log \left(\frac{2P + N_3}{P + N_3} \right). \end{aligned}$$

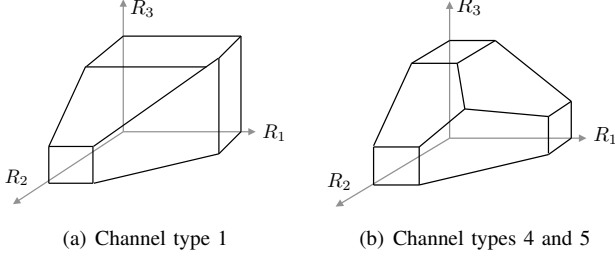


Fig. 2. The shape of the outer bound region. The regions for channel types 2 and 3 look similar to the one for channel type 1 (with change of axis).

Proof:

$$\begin{aligned}
& n(R_1 + R_3 - \epsilon) \\
& \leq I(X_1^n; Y_1^n) + I(X_3^n; Y_3^n) \\
& \leq I(X_1^n; Y_1^n | X_3^n) + I(X_3^n; Y_3^n | X_2^n) \\
& = h(Y_1^n | X_3^n) - h(Y_1^n | X_1^n, X_3^n) \\
& \quad + h(Y_3^n | X_2^n) - h(Y_3^n | X_2^n, X_3^n) \\
& = h(X_1^n + Z_1^n) - h(Z_1^n) \\
& \quad + h(X_1^n + X_3^n + Z_3^n) - h(X_1^n + Z_3^n) \\
& \leq \frac{n}{2} \log \left(\frac{P + N_1}{N_1} \right) + \frac{n}{2} \log \left(\frac{2P + N_3}{P + N_3} \right).
\end{aligned}$$

$$\begin{aligned}
& n(R_2 + R_3 - \epsilon) \\
& \leq I(X_2^n; Y_2^n) + I(X_3^n; Y_3^n) \\
& \leq I(X_2^n; Y_2^n | X_3^n) + I(X_3^n; Y_3^n | X_1^n) \\
& = h(Y_2^n | X_3^n) - h(Y_2^n | X_2^n, X_3^n) \\
& \quad + h(Y_3^n | X_1^n) - h(Y_3^n | X_1^n, X_3^n) \\
& = h(X_2^n + Z_2^n) - h(Z_2^n) \\
& \quad + h(X_2^n + X_3^n + Z_3^n) - h(X_2^n + Z_3^n) \\
& \leq \frac{n}{2} \log \left(\frac{P + N_2}{N_2} \right) + \frac{n}{2} \log \left(\frac{2P + N_3}{P + N_3} \right).
\end{aligned}$$

■

D. Channel Type 4

In this section, we present an outer bound on the capacity region of Type 4 channel defined by

$$\begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} + \begin{bmatrix} Z_1 \\ Z_2 \\ Z_3 \end{bmatrix}.$$

This is a cyclic Gaussian interference channel [6]. We first show that channel type 4 is in the mixed interference regime. By normalizing the noise variances, we get the equivalent channel given by

$$\begin{bmatrix} Y'_1 \\ Y'_2 \\ Y'_3 \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} + \begin{bmatrix} Z'_1 \\ Z'_2 \\ Z'_3 \end{bmatrix}$$

where $Y'_k = \frac{1}{\sqrt{N_k}} Y_k$, $Z'_k = \frac{1}{\sqrt{N_k}} Z_k$, $N_0 = \mathbb{E}[Z_k'^2] = 1$, $\mathbb{E}[X_k'^2] \leq P_k = P$ and

$$\begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{N_1}} & 0 & \frac{1}{\sqrt{N_1}} \\ \frac{1}{\sqrt{N_2}} & \frac{1}{\sqrt{N_2}} & 0 \\ 0 & \frac{1}{\sqrt{N_3}} & \frac{1}{\sqrt{N_3}} \end{bmatrix}.$$

With the usual definitions of $\text{SNR}_k = \frac{h_{kk}^2 P_k}{N_0}$ and

$\text{INR}_k = \frac{h_{jk}^2 P_k}{N_0}$ for $j \neq k$ as in [1], [6],

$$\text{SNR}_1 = \frac{P}{N_1} \geq \text{INR}_1 = \frac{P}{N_2} \quad (4)$$

$$\text{SNR}_2 = \frac{P}{N_2} \geq \text{INR}_2 = \frac{P}{N_3} \quad (5)$$

$$\text{SNR}_3 = \frac{P}{N_3} \leq \text{INR}_3 = \frac{P}{N_1}. \quad (6)$$

We state the outer bound in the following theorem.

Theorem 6: The capacity region of Type 4 channel is contained in the following outer bound region:

$$\begin{aligned}
& R_k \leq C_k, \quad k = 1, 2, 3 \\
& R_1 + R_2 \leq \frac{1}{2} \log \left(1 + \frac{P}{N_1} \right) + \frac{1}{2} \log \left(\frac{2P + N_2}{P + N_2} \right) \\
& R_1 + R_3 \leq \frac{1}{2} \log \left(1 + \frac{2P}{N_1} \right) \\
& R_2 + R_3 \leq \frac{1}{2} \log \left(1 + \frac{P}{N_2} \right) + \frac{1}{2} \log \left(\frac{2P + N_3}{P + N_3} \right).
\end{aligned}$$

Proof:

$$\begin{aligned}
& n(R_1 + R_2 - \epsilon) \\
& \leq I(X_1^n; Y_1^n) + I(X_2^n; Y_2^n) \\
& \leq I(X_1^n; Y_1^n | X_3^n) + I(X_2^n; Y_2^n) \\
& = h(Y_1^n | X_3^n) - h(Y_1^n | X_1^n, X_3^n) \\
& \quad + h(Y_2^n) - h(Y_2^n | X_2^n) \\
& = h(X_1^n + Z_1^n) - h(Z_1^n) \\
& \quad + h(X_1^n + X_2^n + Z_2^n) - h(X_1^n + Z_2^n) \\
& \leq \frac{n}{2} \log \left(\frac{P + N_1}{N_1} \right) + \frac{n}{2} \log \left(\frac{2P + N_2}{P + N_2} \right).
\end{aligned}$$

$$\begin{aligned}
& n(R_2 + R_3 - \epsilon) \\
& \leq I(X_2^n; Y_2^n) + I(X_3^n; Y_3^n) \\
& \leq I(X_2^n; Y_2^n | X_1^n) + I(X_3^n; Y_3^n) \\
& = h(Y_2^n | X_1^n) - h(Y_2^n | X_1^n, X_2^n) \\
& \quad + h(Y_3^n) - h(Y_3^n | X_3^n) \\
& = h(X_2^n + Z_2^n) - h(Z_2^n) \\
& \quad + h(X_2^n + X_3^n + Z_3^n) - h(X_2^n + Z_3^n) \\
& \leq \frac{n}{2} \log \left(\frac{P + N_2}{N_2} \right) + \frac{n}{2} \log \left(\frac{2P + N_3}{P + N_3} \right).
\end{aligned}$$

$$\begin{aligned}
n(R_1 + R_3 - \epsilon) &\leq I(X_1^n; Y_1^n) + I(X_3^n; Y_3^n) \\
&\leq I(X_1^n; Y_1^n) + I(X_3^n; Y_3^n | X_2^n) \\
&\leq I(X_1^n; Y_1^n) + I(X_3^n; Y_1^n | X_1^n) \\
&\leq I(X_1^n, X_3^n; Y_1^n) \\
&= h(Y_1^n) - h(Y_1^n | X_1^n, X_3^n) \\
&= h(X_1^n + X_3^n + Z_1^n) - h(Z_1^n) \\
&\leq \frac{n}{2} \log \left(\frac{2P + N_1}{N_1} \right)
\end{aligned}$$

where we used the fact that $I(X_3^n; Y_3^n | X_2^n) = I(X_3^n; X_3^n + Z_3^n) \leq I(X_3^n; X_3^n + Z_1^n) = I(X_3^n; Y_1^n | X_1^n)$. ■

E. Channel Type 5

In this section, we present an outer bound on the capacity region of Type 5 channel defined by

$$\begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} + \begin{bmatrix} Z_1 \\ Z_2 \\ Z_3 \end{bmatrix}.$$

This is a cyclic Gaussian interference channel [6]. We first show that channel type 5 is in the mixed interference regime. By normalizing the noise variances, we get the equivalent channel given by

$$\begin{bmatrix} Y'_1 \\ Y'_2 \\ Y'_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{N_1}} & \frac{1}{\sqrt{N_1}} & 0 \\ 0 & \frac{1}{\sqrt{N_2}} & \frac{1}{\sqrt{N_2}} \\ \frac{1}{\sqrt{N_3}} & 0 & \frac{1}{\sqrt{N_3}} \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} + \begin{bmatrix} Z'_1 \\ Z'_2 \\ Z'_3 \end{bmatrix}.$$

We can see that

$$\text{SNR}_1 = \frac{P}{N_1} \geq \text{INR}_1 = \frac{P}{N_3} \quad (7)$$

$$\text{SNR}_2 = \frac{P}{N_2} \leq \text{INR}_2 = \frac{P}{N_1} \quad (8)$$

$$\text{SNR}_3 = \frac{P}{N_3} \leq \text{INR}_3 = \frac{P}{N_2}. \quad (9)$$

We state the outer bound in the following theorem.

Theorem 7: The capacity region of Type 5 channel is contained in the following outer bound region:

$$\begin{aligned}
R_k &\leq C_k, \quad k = 1, 2, 3 \\
R_1 + R_2 &\leq \frac{1}{2} \log \left(1 + \frac{2P}{N_1} \right) \\
R_2 + R_3 &\leq \frac{1}{2} \log \left(1 + \frac{2P}{N_2} \right) \\
R_1 + R_3 &\leq \frac{1}{2} \log \left(1 + \frac{P}{N_1} \right) + \frac{1}{2} \log \left(\frac{2P + N_3}{P + N_3} \right).
\end{aligned}$$

Proof:

$$\begin{aligned}
n(R_1 + R_2 - \epsilon) &\leq I(X_1^n; Y_1^n) + I(X_2^n; Y_2^n) \\
&\leq I(X_1^n; Y_1^n) + I(X_2^n; Y_2^n | X_3^n) \\
&\leq I(X_1^n; Y_1^n) + I(X_2^n; Y_1^n | X_1^n) \\
&\leq I(X_1^n, X_2^n; Y_1^n) \\
&= h(Y_1^n) - h(Y_1^n | X_1^n, X_2^n) \\
&= h(X_1^n + X_2^n + Z_1^n) - h(Z_1^n) \\
&\leq \frac{n}{2} \log \left(\frac{2P + N_1}{N_1} \right)
\end{aligned}$$

where we used the fact that $I(X_2^n; Y_2^n | X_3^n) = I(X_2^n; X_2^n + Z_2^n) \leq I(X_2^n; X_2^n + Z_1^n) = I(X_2^n; Y_1^n | X_1^n)$.

$$\begin{aligned}
n(R_2 + R_3 - \epsilon) &\leq I(X_2^n; Y_2^n) + I(X_3^n; Y_3^n) \\
&\leq I(X_2^n; Y_2^n) + I(X_3^n; Y_3^n | X_1^n) \\
&\leq I(X_2^n; Y_2^n) + I(X_3^n; Y_2^n | X_2^n) \\
&\leq I(X_2^n, X_3^n; Y_2^n) \\
&= h(Y_2^n) - h(Y_2^n | X_2^n, X_3^n) \\
&= h(X_2^n + X_3^n + Z_2^n) - h(Z_2^n) \\
&\leq \frac{n}{2} \log \left(\frac{2P + N_2}{N_2} \right)
\end{aligned}$$

where we used the fact that $I(X_3^n; Y_3^n | X_1^n) = I(X_3^n; X_3^n + Z_3^n) \leq I(X_3^n; X_3^n + Z_2^n) = I(X_3^n; Y_2^n | X_2^n)$.

$$\begin{aligned}
n(R_1 + R_3 - \epsilon) &\leq I(X_1^n; Y_1^n) + I(X_3^n; Y_3^n) \\
&\leq I(X_1^n; Y_1^n | X_2^n) + I(X_3^n; Y_3^n) \\
&= h(Y_1^n | X_2^n) - h(Y_1^n | X_1^n, X_2^n) \\
&\quad + h(Y_3^n) - h(Y_3^n | X_3^n) \\
&= h(X_1^n + Z_1^n) - h(Z_1^n) \\
&\quad + h(X_1^n + X_3^n + Z_3^n) - h(X_1^n + Z_3^n) \\
&\leq \frac{n}{2} \log \left(\frac{P + N_1}{N_1} \right) + \frac{n}{2} \log \left(\frac{2P + N_3}{P + N_3} \right)
\end{aligned}$$

■

F. Relaxed Outer Bounds

For ease of gap calculation, we also derive relaxed outer bounds. First, we can see that for $N_j \leq N_k$,

$$\frac{1}{2} \log \left(1 + \frac{P}{N_j} \right) + \frac{1}{2} \log \left(\frac{2P + N_k}{P + N_k} \right) \leq \frac{1}{2} \log \left(1 + \frac{2P}{N_j} \right).$$

Five outer bound theorems in this section, together with this inequality, give the sum-rate bound expression in Theorem 1.

Next, we can assume that $P \geq 3N_j$ for $j = 1, 2, 3$. Otherwise, showing one-bit gap capacity is trivial as the

Type	Outer bound region \mathcal{R}_o	Relaxed outer bound region \mathcal{R}'_o	Two-dimensional cross-section of \mathcal{R}'_o
1	$R_k \leq C_k, k = 1, 2, 3$ $R_1 + R_2 \leq \frac{1}{2} \log \left(\frac{P+N_1}{N_1} \cdot \frac{2P+N_2}{P+N_2} \right)$ $R_2 + R_3 \leq \frac{1}{2} \log \left(\frac{P+N_2}{N_2} \cdot \frac{2P+N_3}{P+N_3} \right)$	$R_k \leq \frac{1}{2} \log \left(\frac{P}{N_k} \cdot \frac{4}{3} \right)$ $R_1 + R_2 \leq \frac{1}{2} \log \left(\frac{P}{N_1} \cdot \frac{7}{3} \right)$ $R_2 + R_3 \leq \frac{1}{2} \log \left(\frac{P}{N_2} \cdot \frac{7}{3} \right)$	At some $R_2 \in [0, C_2]$, $R_1 \leq \min \left\{ \frac{1}{2} \log \left(\frac{P}{N_1} \cdot \frac{7}{3} \right) - R_2, \frac{1}{2} \log \left(\frac{P}{N_1} \cdot \frac{4}{3} \right) \right\}$ $R_3 \leq \min \left\{ \frac{1}{2} \log \left(\frac{P}{N_2} \cdot \frac{7}{3} \right) - R_2, \frac{1}{2} \log \left(\frac{P}{N_3} \cdot \frac{4}{3} \right) \right\}$
2	$R_k \leq C_k, k = 1, 2, 3$ $R_1 + R_2 \leq \frac{1}{2} \log \left(\frac{P+N_1}{N_1} \cdot \frac{2P+N_2}{P+N_2} \right)$ $R_1 + R_3 \leq \frac{1}{2} \log \left(\frac{P+N_1}{N_1} \cdot \frac{2P+N_3}{P+N_3} \right)$	$R_k \leq \frac{1}{2} \log \left(\frac{P}{N_k} \cdot \frac{4}{3} \right)$ $R_1 + R_2 \leq \frac{1}{2} \log \left(\frac{P}{N_1} \cdot \frac{7}{3} \right)$ $R_1 + R_3 \leq \frac{1}{2} \log \left(\frac{P}{N_1} \cdot \frac{7}{3} \right)$	At some $R_1 \in [0, C_1]$, $R_2 \leq \min \left\{ \frac{1}{2} \log \left(\frac{P}{N_1} \cdot \frac{7}{3} \right) - R_1, \frac{1}{2} \log \left(\frac{P}{N_2} \cdot \frac{4}{3} \right) \right\}$ $R_3 \leq \min \left\{ \frac{1}{2} \log \left(\frac{P}{N_1} \cdot \frac{7}{3} \right) - R_1, \frac{1}{2} \log \left(\frac{P}{N_3} \cdot \frac{4}{3} \right) \right\}$
3	$R_k \leq C_k, k = 1, 2, 3$ $R_1 + R_3 \leq \frac{1}{2} \log \left(\frac{P+N_1}{N_1} \cdot \frac{2P+N_3}{P+N_3} \right)$ $R_2 + R_3 \leq \frac{1}{2} \log \left(\frac{P+N_2}{N_2} \cdot \frac{2P+N_3}{P+N_3} \right)$	$R_k \leq \frac{1}{2} \log \left(\frac{P}{N_k} \cdot \frac{4}{3} \right)$ $R_1 + R_3 \leq \frac{1}{2} \log \left(\frac{P}{N_1} \cdot \frac{7}{3} \right)$ $R_2 + R_3 \leq \frac{1}{2} \log \left(\frac{P}{N_2} \cdot \frac{7}{3} \right)$	At some $R_3 \in [0, C_3]$, $R_1 \leq \min \left\{ \frac{1}{2} \log \left(\frac{P}{N_1} \cdot \frac{7}{3} \right) - R_3, \frac{1}{2} \log \left(\frac{P}{N_1} \cdot \frac{4}{3} \right) \right\}$ $R_2 \leq \min \left\{ \frac{1}{2} \log \left(\frac{P}{N_2} \cdot \frac{7}{3} \right) - R_3, \frac{1}{2} \log \left(\frac{P}{N_2} \cdot \frac{4}{3} \right) \right\}$
4	$R_k \leq C_k, k = 1, 2, 3$ $R_1 + R_2 \leq \frac{1}{2} \log \left(\frac{P+N_1}{N_1} \cdot \frac{2P+N_2}{P+N_2} \right)$ $R_1 + R_3 \leq \frac{1}{2} \log \left(\frac{2P+N_1}{N_1} \right)$ $R_2 + R_3 \leq \frac{1}{2} \log \left(\frac{P+N_2}{N_2} \cdot \frac{2P+N_3}{P+N_3} \right)$	$R_k \leq \frac{1}{2} \log \left(\frac{P}{N_k} \cdot \frac{4}{3} \right)$ $R_1 + R_2 \leq \frac{1}{2} \log \left(\frac{P}{N_1} \cdot \frac{7}{3} \right)$ $R_1 + R_3 \leq \frac{1}{2} \log \left(\frac{P}{N_1} \cdot \frac{7}{3} \right)$ $R_2 + R_3 \leq \frac{1}{2} \log \left(\frac{P}{N_2} \cdot \frac{7}{3} \right)$	At some $R_1 \in [0, C_1]$, $R_2 \leq \min \left\{ \frac{1}{2} \log \left(\frac{P}{N_1} \cdot \frac{7}{3} \right) - R_1, \frac{1}{2} \log \left(\frac{P}{N_2} \cdot \frac{4}{3} \right) \right\}$ $R_3 \leq \min \left\{ \frac{1}{2} \log \left(\frac{P}{N_1} \cdot \frac{7}{3} \right) - R_1, \frac{1}{2} \log \left(\frac{P}{N_3} \cdot \frac{4}{3} \right) \right\}$ $R_2 + R_3 \leq \frac{1}{2} \log \left(\frac{P}{N_2} \cdot \frac{7}{3} \right)$
5	$R_k \leq C_k, k = 1, 2, 3$ $R_1 + R_2 \leq \frac{1}{2} \log \left(\frac{2P+N_1}{N_1} \right)$ $R_2 + R_3 \leq \frac{1}{2} \log \left(\frac{2P+N_2}{N_2} \right)$ $R_1 + R_3 \leq \frac{1}{2} \log \left(\frac{P+N_1}{N_1} \cdot \frac{2P+N_3}{P+N_3} \right)$	$R_k \leq \frac{1}{2} \log \left(\frac{P}{N_k} \cdot \frac{4}{3} \right)$ $R_1 + R_2 \leq \frac{1}{2} \log \left(\frac{P}{N_1} \cdot \frac{7}{3} \right)$ $R_2 + R_3 \leq \frac{1}{2} \log \left(\frac{P}{N_2} \cdot \frac{7}{3} \right)$ $R_1 + R_3 \leq \frac{1}{2} \log \left(\frac{P}{N_1} \cdot \frac{7}{3} \right)$	At some $R_2 \in [0, C_2]$, $R_1 \leq \min \left\{ \frac{1}{2} \log \left(\frac{P}{N_1} \cdot \frac{7}{3} \right) - R_2, \frac{1}{2} \log \left(\frac{P}{N_1} \cdot \frac{4}{3} \right) \right\}$ $R_3 \leq \min \left\{ \frac{1}{2} \log \left(\frac{P}{N_2} \cdot \frac{7}{3} \right) - R_2, \frac{1}{2} \log \left(\frac{P}{N_3} \cdot \frac{4}{3} \right) \right\}$ $R_1 + R_3 \leq \frac{1}{2} \log \left(\frac{P}{N_1} \cdot \frac{7}{3} \right)$

TABLE II
CAPACITY OUTER BOUNDS

capacity region is included in the unit hypercube, i.e., $R_j \leq \frac{1}{2} \log \left(1 + \frac{P}{N_j} \right) < 1$. For $P \geq 3N_j$,

$$\begin{aligned}
 \frac{1}{2} \log \left(1 + \frac{2P}{N_j} \right) &= \frac{1}{2} \log \left(\frac{P}{N_j} \right) + \frac{1}{2} \log \left(\frac{N_j}{P} + 2 \right) \\
 &\leq \frac{1}{2} \log \left(\frac{P}{N_j} \right) + \frac{1}{2} \log \left(\frac{7}{3} \right) \\
 \frac{1}{2} \log \left(1 + \frac{P}{N_j} \right) &\leq \frac{1}{2} \log \left(\frac{P}{N_j} \right) + \frac{1}{2} \log \left(\frac{4}{3} \right).
 \end{aligned}$$

The resulting relaxed outer bounds \mathcal{R}'_o are summarized in Table II.

III. INNER BOUND: CHANNEL TYPE 1

Theorem 8: Given $\alpha = (\alpha_0, \alpha_2) \in [0, 1]^2$, the rate region \mathcal{R}_α is defined by

$$\begin{aligned}
 R_1 &\leq \frac{1}{2} \log^+ \left(\frac{1 - \alpha_0}{2 - \alpha_0} + \frac{(1 - \alpha_0)P}{(\alpha_0 + \alpha_2)P + N_2} \right) \\
 &\quad + \frac{1}{2} \log \left(1 + \frac{\alpha_0 P}{N_1} \right) \\
 R_2 &\leq \frac{1}{2} \log \left(1 + \frac{\alpha_2 P}{\alpha_0 P + N_2} \right) \\
 R_3 &\leq \frac{1}{2} \log^+ \left(\frac{1}{2 - \alpha_0} + \frac{P}{(\alpha_0 + \alpha_2)P + N_3} \right)
 \end{aligned}$$

where $\log^+(\cdot) = \max\{0, \log(\cdot)\}$. And,

$$\mathcal{R} = \text{CONV} \left(\bigcup_{\alpha} \mathcal{R}_\alpha \right)$$

is achievable where $\text{CONV}(\cdot)$ is convex hull operator.

A. Preliminaries: Lattice Coding

Lattice Λ is a discrete subgroup of \mathbb{R}^n , $\Lambda = \{\mathbf{t} = \mathbf{G}\mathbf{u} : \mathbf{u} \in \mathbb{Z}^n\}$ where $\mathbf{G} \in \mathbb{R}^{n \times n}$ is a real generator matrix. Quantization with respect to Λ is $Q_\Lambda(\mathbf{x}) = \arg \min_{\lambda \in \Lambda} \|\mathbf{x} - \lambda\|$. Modulo operation with respect to Λ is $M_\Lambda(\mathbf{x}) = [\mathbf{x}] \bmod \Lambda = \mathbf{x} - Q_\Lambda(\mathbf{x})$. For convenience, we use both notations $M_\Lambda(\cdot)$ and $[\cdot] \bmod \Lambda$ interchangeably. Fundamental Voronoi region of Λ is $\mathcal{V}(\Lambda) = \{\mathbf{x} : Q_\Lambda(\mathbf{x}) = \mathbf{0}\}$. Volume of the Voronoi region of Λ is $V(\Lambda) = \int_{\mathcal{V}(\Lambda)} d\mathbf{x}$. Normalized second moment of Λ is $G(\Lambda) = \frac{\sigma^2(\Lambda)}{V(\Lambda)^{2/n}}$ where $\sigma^2(\Lambda) = \frac{1}{nV(\Lambda)} \int_{\mathcal{V}(\Lambda)} \|\mathbf{x}\|^2 d\mathbf{x}$. Lattices Λ_1, Λ_2 and Λ are said to be nested if $\Lambda \subseteq \Lambda_2 \subseteq \Lambda_1$. For nested lattices $\Lambda_2 \subset \Lambda_1$, $\Lambda_1/\Lambda_2 = \Lambda_1 \cap \mathcal{V}(\Lambda_2)$.

We briefly review the lattice decoding procedure in [12]. We use nested lattices $\Lambda \subseteq \Lambda_t$ with $\sigma^2(\Lambda) = S$, $G(\Lambda) = \frac{1}{2\pi e}$, and $V(\Lambda) = (2\pi e S)^{\frac{n}{2}}$. The transmitter sends $\mathbf{x} = [\mathbf{t} + \mathbf{d}] \bmod \Lambda$ over the point-to-point Gaussian channel $\mathbf{y} = \mathbf{x} + \mathbf{z}$ where the codeword $\mathbf{t} \in \Lambda_t \cap \mathcal{V}(\Lambda)$, the dither signal $\mathbf{d} \sim \text{Unif}(\mathcal{V}(\Lambda))$, the transmit power $\frac{1}{n} \|\mathbf{x}\|^2 = S$ and the noise $\mathbf{z} \sim \mathcal{N}(0, N\mathbf{I})$. The code rate is given by $R = \frac{1}{n} \log \left(\frac{V(\Lambda)}{V(\Lambda_t)} \right)$.

After linear scaling, dither removal, and mod- Λ operation, we get

$$\mathbf{y}' = [\beta \mathbf{y} - \mathbf{d}] \bmod \Lambda = [\mathbf{t} + \mathbf{z}_e] \bmod \Lambda \quad (10)$$

where the effective noise is $\mathbf{z}_e = (\beta - 1)\mathbf{x} + \beta \mathbf{z}_1$ and its variance $\sigma_e^2 = \frac{1}{n} \mathbb{E}[\|\mathbf{z}_e\|^2] = (\beta - 1)^2 S + \beta^2 N$. With the MMSE scaling factor $\beta = \frac{S}{S+N}$ plugged in, we get $\sigma_e^2 = \beta N = \frac{SN}{S+N}$. The capacity of the mod- Λ channel [12] between

\mathbf{t} and \mathbf{y} is

$$\begin{aligned}
\frac{1}{n}I(\mathbf{t}; \mathbf{y}) &= \frac{1}{n}h(\mathbf{y}) - \frac{1}{n}h(\mathbf{y}|\mathbf{t}) \\
&= \frac{1}{n}h(\mathbf{y}) - \frac{1}{n}h(\mathbf{z} \bmod \Lambda) \\
&\geq \frac{1}{n}h(\mathbf{y}) - \frac{1}{n}h(\mathbf{z}) \\
&= \frac{1}{n}\log V(\Lambda) - \frac{1}{n}h(\mathbf{z}) \\
&= \frac{1}{2}\log\left(\frac{S}{\beta N}\right) \\
&= \frac{1}{2}\log\left(1 + \frac{S}{N}\right) \\
&= C
\end{aligned}$$

where $I(\cdot)$ and $h(\cdot)$ are mutual information and differential entropy, respectively. For reliable decoding of \mathbf{t} , we have the code rate constraint $R \leq C$. With the choice of lattice parameters, $\sigma^2(\Lambda_t) \geq \beta N$, $G(\Lambda_t) = \frac{1}{2\pi e}$ and $V(\Lambda_t)^{\frac{n}{2}} = \frac{\sigma^2(\Lambda_t)}{G(\Lambda_t)} \geq 2\pi e\beta N$,

$$\begin{aligned}
R &= \frac{1}{n}\log\left(\frac{V(\Lambda)}{V(\Lambda_t)}\right) \\
&\leq \frac{1}{n}\log\left(\frac{(2\pi eS)^{\frac{n}{2}}}{(2\pi e\beta N)^{\frac{n}{2}}}\right) \\
&= \frac{1}{2}\log\left(\frac{S}{\beta N}\right).
\end{aligned}$$

Thus, the constraint $R \leq C$ can be satisfied. By *lattice decoding* [12], we can recover \mathbf{t} , i.e.,

$$Q_{\Lambda_t}(\mathbf{y}') = \mathbf{t}, \quad (11)$$

with probability $1 - P_e$ where

$$P_e = \Pr[Q_{\Lambda_t}(\mathbf{y}') \neq \mathbf{t}] \quad (12)$$

is the probability of decoding error. If we choose Λ to be Poltyrev-good [27], then $P_e \rightarrow 0$ as $n \rightarrow \infty$.

B. Achievable Scheme

We present an achievable scheme for the proof of *Theorem 8*. The achievable scheme is based on rate-splitting, lattice coding, and interference alignment. Message $M_1 \in \{1, 2, \dots, 2^{nR_1}\}$ is split into two parts: $M_{11} \in \{1, 2, \dots, 2^{nR_{11}}\}$ and $M_{10} \in \{1, 2, \dots, 2^{nR_{10}}\}$, so $R_1 = R_{11} + R_{10}$. Transmitter 1 sends $\mathbf{x}_1 = \mathbf{x}_{11} + \mathbf{x}_{10}$ where \mathbf{x}_{11} and \mathbf{x}_{10} are coded signals of M_{11} and M_{10} , respectively. Transmitters 2 and 3 send \mathbf{x}_2 and \mathbf{x}_3 , coded signals of $M_2 \in \{1, 2, \dots, 2^{nR_2}\}$ and $M_3 \in \{1, 2, \dots, 2^{nR_3}\}$. In particular, \mathbf{x}_{11} and \mathbf{x}_3 are lattice-coded signals.

We use the lattice construction of [14], [15] with the lattice partition chain $\Lambda_c/\Lambda_1/\Lambda_3$, so $\Lambda_3 \subset \Lambda_1 \subset \Lambda_c$ are nested lattices. Λ_c is the coding lattice for both \mathbf{x}_{11} and \mathbf{x}_3 . Λ_1 and Λ_3 are shaping lattices for \mathbf{x}_{11} and \mathbf{x}_3 , respectively. The lattice signals are formed by

$$\mathbf{x}_{11} = [\mathbf{t}_{11} + \mathbf{d}_{11}] \bmod \Lambda_1 \quad (13)$$

$$\mathbf{x}_3 = [\mathbf{t}_3 + \mathbf{d}_3] \bmod \Lambda_3 \quad (14)$$

where $\mathbf{t}_{11} \in \Lambda_c \cap \mathcal{V}(\Lambda_1)$ and $\mathbf{t}_3 \in \Lambda_c \cap \mathcal{V}(\Lambda_3)$ are lattice codewords. The dither signals \mathbf{d}_{11} and \mathbf{d}_3 are uniformly distributed over $\mathcal{V}(\Lambda_1)$ and $\mathcal{V}(\Lambda_3)$, respectively. To satisfy power constraints, we choose $\mathbb{E}[\|\mathbf{x}_{11}\|^2] = n\sigma^2(\Lambda_1) = (1 - \alpha_1)nP$, $\mathbb{E}[\|\mathbf{x}_{10}\|^2] = \alpha_1nP$, $\mathbb{E}[\|\mathbf{x}_2\|^2] = \alpha_2nP$, $\mathbb{E}[\|\mathbf{x}_3\|^2] = n\sigma^2(\Lambda_3) = nP$.

With the choice of transmit signals, the received signals are given by

$$\begin{aligned}
\mathbf{y}_1 &= \mathbf{x}_{11} + \mathbf{x}_2 + \mathbf{x}_{10} + \mathbf{z}_1 \\
\mathbf{y}_2 &= [\mathbf{x}_{11} + \mathbf{x}_3] + \mathbf{x}_2 + \mathbf{z}'_2 \\
\mathbf{y}_3 &= \mathbf{x}_3 + \mathbf{z}'_3,
\end{aligned}$$

where $\mathbf{x}_f = [\mathbf{x}_{11} + \mathbf{x}_3]$ is the sum of interference, and $\mathbf{z}'_2 = \mathbf{x}_{10} + \mathbf{z}_2$ and $\mathbf{z}'_3 = \mathbf{x}_2 + \mathbf{z}_3$ are the effective Gaussian noise. The signal scale diagram at each receiver is shown in Fig. 3 (a).

At the receivers, successive decoding is performed in the following order: $\mathbf{x}_{11} \rightarrow \mathbf{x}_2 \rightarrow \mathbf{x}_{10}$ at receiver 1, $\mathbf{x}_f \rightarrow \mathbf{x}_2$ at receiver 2, and receiver 3 only decodes \mathbf{x}_3 .

Note that the aligned lattice codewords $\mathbf{t}_{11} + \mathbf{t}_3 \in \Lambda_c$, and $\mathbf{t}_f = [\mathbf{t}_{11} + \mathbf{t}_3] \bmod \Lambda_1 \in \Lambda_c \cap \mathcal{V}(\Lambda_1)$. We state the relationship between \mathbf{x}_f and \mathbf{t}_f in the following lemmas.

Lemma 1: The following holds.

$$[\mathbf{x}_f - \mathbf{d}_f] \bmod \Lambda_1 = \mathbf{t}_f$$

where $\mathbf{d}_f = \mathbf{d}_{11} + \mathbf{d}_3$.

Proof:

$$\begin{aligned}
&[\mathbf{x}_f - \mathbf{d}_f] \bmod \Lambda_1 \\
&= [M_{\Lambda_1}(\mathbf{t}_{11} + \mathbf{d}_{11}) + M_{\Lambda_3}(\mathbf{t}_3 + \mathbf{d}_3) - \mathbf{d}_f] \bmod \Lambda_1 \\
&= [M_{\Lambda_1}(\mathbf{t}_{11} + \mathbf{d}_{11}) + M_{\Lambda_1}(\mathbf{t}_3 + \mathbf{d}_3) - \mathbf{d}_f] \bmod \Lambda_1 \\
&= [\mathbf{t}_{11} + \mathbf{d}_{11} + \mathbf{t}_3 + \mathbf{d}_3 - \mathbf{d}_f] \bmod \Lambda_1 \\
&= [\mathbf{t}_{11} + \mathbf{t}_3] \bmod \Lambda_1 \\
&= \mathbf{t}_f
\end{aligned}$$

The second and third equalities are due to distributive law and the identity in the following lemma. ■

Lemma 2: For any nested lattices $\Lambda_3 \subset \Lambda_1$ and any $\mathbf{x} \in \mathbb{R}^n$, it holds that

$$[M_{\Lambda_3}(\mathbf{x})] \bmod \Lambda_1 = [\mathbf{x}] \bmod \Lambda_1.$$

Proof:

$$\begin{aligned}
&[M_{\Lambda_3}(\mathbf{x})] \bmod \Lambda_1 \\
&= [\mathbf{x} - \lambda_3] \bmod \Lambda_1 \\
&= [M_{\Lambda_1}(\mathbf{x}) - M_{\Lambda_1}(\lambda_3)] \bmod \Lambda_1 \\
&= [M_{\Lambda_1}(\mathbf{x}) - \lambda_3 + Q_{\Lambda_1}(\lambda_3)] \bmod \Lambda_1 \\
&= [M_{\Lambda_1}(\mathbf{x})] \bmod \Lambda_1 \\
&= [\mathbf{x}] \bmod \Lambda_1
\end{aligned}$$

where $\lambda_3 = Q_{\Lambda_3}(\mathbf{x}) \in \Lambda_1$, thus $Q_{\Lambda_1}(\lambda_3) = \lambda_3$. ■

Lemma 3: The following holds.

$$[\mathbf{t}_f + \mathbf{d}_f] \bmod \Lambda_1 = [\mathbf{x}_f] \bmod \Lambda_1.$$

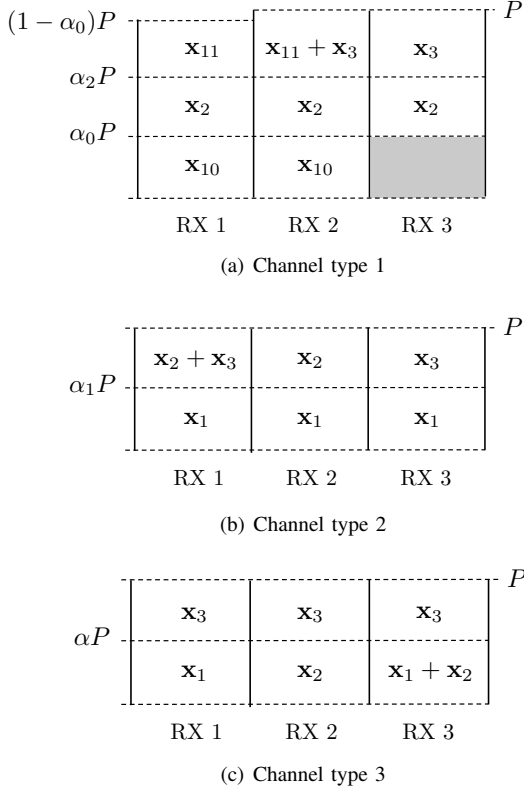


Fig. 3. Signal scale diagram.

Proof:

$$\begin{aligned}
& [\mathbf{t}_f + \mathbf{d}_f] \bmod \Lambda_1 \\
&= [M_{\Lambda_1}(\mathbf{t}_{11} + \mathbf{t}_3) + \mathbf{d}_f] \bmod \Lambda_1 \\
&= [\mathbf{t}_{11} + \mathbf{t}_3 + \mathbf{d}_f] \bmod \Lambda_1 \\
&= [M_{\Lambda_1}(\mathbf{t}_{11} + \mathbf{d}_{11}) + M_{\Lambda_1}(\mathbf{t}_3 + \mathbf{d}_3)] \bmod \Lambda_1 \\
&= [M_{\Lambda_1}(\mathbf{t}_{11} + \mathbf{d}_{11}) + M_{\Lambda_3}(\mathbf{t}_3 + \mathbf{d}_3)] \bmod \Lambda_1 \\
&= [\mathbf{x}_{11} + \mathbf{x}_3] \bmod \Lambda_1 \\
&= [\mathbf{x}_f] \bmod \Lambda_1
\end{aligned}$$

Receiver 2 does not need to recover the codewords \mathbf{t}_{11} and \mathbf{t}_3 but the real sum \mathbf{x}_f to remove the interference from \mathbf{y}_2 . Since $\mathbf{x}_f = M_{\Lambda_1}(\mathbf{x}_f) + Q_{\Lambda_1}(\mathbf{x}_f)$, we first recover the modulo part and then the quantized part to cancel out \mathbf{x}_f . This idea appeared in [17] as an achievable scheme for the many-to-one interference channel.

The mod- Λ_1 channel between \mathbf{t}_f and \mathbf{y}'_2 is given by

$$\mathbf{y}'_2 = [\beta_2 \mathbf{y}_2 - \mathbf{d}_f] \bmod \Lambda_1 \quad (15)$$

$$= [\mathbf{x}_f - \mathbf{d}_f + \mathbf{z}_{e2}] \bmod \Lambda_1 \quad (16)$$

$$= [\mathbf{t}_f + \mathbf{z}_{e2}] \bmod \Lambda_1 \quad (17)$$

where the effective noise $\mathbf{z}_{e2} = (\beta_2 - 1)\mathbf{x}_f + \beta_2(\mathbf{x}_2 + \mathbf{x}_{10} + \mathbf{z}_2)$. Note that $\mathbb{E}[\|\mathbf{x}_f\|^2] = (\bar{\alpha}_0 + 1)nP$, and the effective noise variance $\sigma_{e2}^2 = \frac{1}{n}\mathbb{E}[\|\mathbf{z}_{e2}\|^2] = (\beta_2 - 1)^2(\bar{\alpha}_0 + 1)P + \beta_2^2 N_{e2}$ where $N_{e2} = (\alpha_0 + \alpha_2)P + N_2$. With the MMSE scaling factor $\beta_2 = \frac{(\bar{\alpha}_0 + 1)P}{(\bar{\alpha}_0 + 1)P + N_{e2}}$ plugged in, we get $\sigma_{e2}^2 = \beta_2 N_{e2} =$

$\frac{(\bar{\alpha}_0 + 1)PN_{e2}}{(\bar{\alpha}_0 + 1)P + N_{e2}}$. The capacity of the mod- Λ_1 channel between \mathbf{t}_f and \mathbf{y}'_2 is

$$\begin{aligned}
& \frac{1}{n} I(\mathbf{t}_f; \mathbf{y}'_2) \\
& \geq \frac{1}{n} \log \left(\frac{V(\Lambda_1)}{2^{h(\mathbf{z}_{e2})}} \right) \\
& = \frac{1}{2} \log \left(\frac{\bar{\alpha}_0 P}{\beta_2 N_{e2}} \right) \\
& = \frac{1}{2} \log \left(\frac{\bar{\alpha}_0(\bar{\alpha}_0 + 1)P + \bar{\alpha}_0 N_{e2}}{(\bar{\alpha}_0 + 1)N_{e2}} \right) \\
& = \frac{1}{2} \log \left(\frac{\bar{\alpha}_0}{\bar{\alpha}_0 + 1} + \frac{\bar{\alpha}_0 P}{N_{e2}} \right) \\
& = \frac{1}{2} \log \left(\frac{\bar{\alpha}_0}{\bar{\alpha}_0 + 1} + \frac{\bar{\alpha}_0 P}{(\alpha_0 + \alpha_2)P + N_2} \right) \\
& = C_f
\end{aligned}$$

For reliable decoding of \mathbf{t}_f at receiver 2, we have the code rate constraint $R_{11} = \frac{1}{n} \log \left(\frac{V(\Lambda_1)}{V(\Lambda_c)} \right) \leq C_f$. This also implies that $R_3 = \frac{1}{n} \log \left(\frac{V(\Lambda_2)}{V(\Lambda_c)} \right) \leq C_f + \frac{1}{n} \log \left(\frac{V(\Lambda_2)}{V(\Lambda_1)} \right) = \frac{1}{2} \log \left(\frac{P}{\beta_2 N_{e2}} \right) = \frac{1}{2} \log \left(\frac{1}{\bar{\alpha}_0 + 1} + \frac{P}{(\alpha_0 + \alpha_2)P + N_2} \right)$. By lattice decoding, we can recover the modulo sum of interference codewords \mathbf{t}_f from \mathbf{y}'_2 . Then, we can recover the real sum \mathbf{x}_f in the following way.

- Recover $M_{\Lambda_1}(\mathbf{x}_f)$ by calculating $[\mathbf{t}_f + \mathbf{d}_f] \bmod \Lambda_1$ (lemma 3).
- Subtract it from the received signal,

$$\mathbf{y}_2 - M_{\Lambda_1}(\mathbf{x}_f) = Q_{\Lambda_1}(\mathbf{x}_f) + \mathbf{z}_2'' \quad (18)$$

where $\mathbf{z}_2'' = \mathbf{x}_2 + \mathbf{x}_{10} + \mathbf{z}_2$.

- Quantize it to recover $Q_{\Lambda_1}(\mathbf{x}_f)$,

$$Q_{\Lambda_1}(Q_{\Lambda_1}(\mathbf{x}_f) + \mathbf{z}_2'') = Q_{\Lambda_1}(\mathbf{x}_f) \quad (19)$$

with probability $1 - P_e$ where

$$P_e = \Pr[Q_{\Lambda_1}(Q_{\Lambda_1}(\mathbf{x}_f) + \mathbf{z}_2'') \neq Q_{\Lambda_1}(\mathbf{x}_f)] \quad (20)$$

is the probability of decoding error. If we choose Λ_1 to be simultaneously Rogers-good and Poltyrev-good [27] with $V(\Lambda_1) \geq V(\Lambda_c)$, then $P_e \rightarrow 0$ as $n \rightarrow \infty$.

- Recover \mathbf{x}_f by adding two vectors,

$$M_{\Lambda_1}(\mathbf{x}_f) + Q_{\Lambda_1}(\mathbf{x}_f) = \mathbf{x}_f. \quad (21)$$

We now proceed to decoding \mathbf{x}_2 from $\mathbf{y}_2 - \mathbf{x}_f = \mathbf{x}_2 + \mathbf{z}_2'$. Since \mathbf{x}_2 is a codeword from an i.i.d. random code for point-to-point channel, we can achieve rate up to

$$R_2 \leq \frac{1}{2} \log \left(\frac{\alpha_2 P}{\alpha_0 P + N_2} \right). \quad (22)$$

At receiver 1, we first decode \mathbf{x}_{11} while treating other signals $\mathbf{x}_2 + \mathbf{x}_{10} + \mathbf{z}_1$ as noise. The effective noise in the mod- Λ_1 channel is $\mathbf{z}_{e1} = (\beta_1 - 1)\mathbf{x}_{11} + \beta_1(\mathbf{x}_2 + \mathbf{x}_{10} + \mathbf{z}_1)$ with variance $\sigma_{e1}^2 = \frac{1}{n}\mathbb{E}[\|\mathbf{z}_{e1}\|^2] = (\beta_1 - 1)^2 \bar{\alpha}_0 P + \beta_1^2 N_{e1}$ where $N_{e1} = (\alpha_0 + \alpha_2)P + N_1$. For reliable decoding, the rate R_{11} must satisfy

$$R_{11} \leq \frac{1}{2} \log \left(\frac{\sigma^2(\Lambda_1)}{\beta_1 \sigma_{e1}^2} \right) = \frac{1}{2} \log \left(1 + \frac{\bar{\alpha}_0 P}{(\alpha_0 + \alpha_2)P + N_1} \right)$$

where the MMSE scaling parameter $\beta_1 = \frac{\alpha_0 P}{\alpha_0 P + N_{e1}}$. Similarly, we have the other rate constraints at receiver 1:

$$R_2 \leq \frac{1}{2} \log \left(1 + \frac{\alpha_2 P}{\alpha_0 P + N_1} \right) \quad (23)$$

$$R_{10} \leq \frac{1}{2} \log \left(1 + \frac{\alpha_0 P}{N_1} \right). \quad (24)$$

At receiver 3, the signal \mathbf{x}_3 is decoded with the effective noise $\mathbf{x}_2 + \mathbf{z}_3$. For reliable decoding, R_3 must satisfy

$$R_3 \leq \frac{1}{2} \log \left(1 + \frac{P}{\alpha_2 P + N_3} \right). \quad (25)$$

In summary,

- \mathbf{x}_{11} decoded at receivers 1 and 2

$$R_{11} \leq T'_{11} = \frac{1}{2} \log \left(1 + \frac{(1 - \alpha_0)P}{(\alpha_0 + \alpha_2)P + N_1} \right)$$

$$R_{11} \leq T''_{11} = \frac{1}{2} \log \left(c_{11} + \frac{(1 - \alpha_0)P}{(\alpha_0 + \alpha_2)P + N_2} \right)$$

where $c_{11} = \frac{(1 - \alpha_0)P}{(1 - \alpha_0)P + P} = \frac{1 - \alpha_0}{2 - \alpha_0}$.

- \mathbf{x}_{10} decoded at receiver 1

$$R_{10} \leq T_{10} = \frac{1}{2} \log \left(1 + \frac{\alpha_0 P}{N_1} \right) \quad (26)$$

- \mathbf{x}_2 decoded at receivers 1 and 2

$$R_2 \leq T'_2 = \frac{1}{2} \log \left(1 + \frac{\alpha_2 P}{\alpha_0 P + N_1} \right) \quad (27)$$

$$R_2 \leq T''_2 = \frac{1}{2} \log \left(1 + \frac{\alpha_2 P}{\alpha_0 P + N_2} \right) \quad (28)$$

- \mathbf{x}_3 decoded at receivers 2 and 3

$$R_3 \leq T'_3 = \frac{1}{2} \log \left(c_3 + \frac{P}{(\alpha_0 + \alpha_2)P + N_2} \right)$$

$$R_3 \leq T''_3 = \frac{1}{2} \log \left(1 + \frac{P}{\alpha_2 P + N_3} \right) \quad (29)$$

where $c_3 = \frac{P}{(1 - \alpha_0)P + P} = \frac{1}{2 - \alpha_0}$.

Note that $0 \leq c_{11} \leq \frac{1}{2}$, $c_{11} + c_3 = 1$, and $\frac{1}{2} \leq c_3 \leq 1$. Putting together, we can see that the following rate region is achievable.

$$R_1 \leq T_1 = \min\{T'_{11}, T''_{11}\} + T_{10} = T''_{11} + T_{10}$$

$$R_2 \leq T_2 = \min\{T'_2, T''_2\} = T'_2$$

$$R_3 \leq T_3 = \min\{T'_3, T''_3\}$$

where

$$T_1 = \frac{1}{2} \log \left(c_{11} + \frac{(1 - \alpha_0)P}{(\alpha_0 + \alpha_2)P + N_2} \right) + \frac{1}{2} \log \left(1 + \frac{\alpha_0 P}{N_1} \right) \quad (30)$$

$$T_2 = \frac{1}{2} \log \left(1 + \frac{\alpha_2 P}{\alpha_0 P + N_2} \right) \quad (31)$$

$$T_3 \geq \frac{1}{2} \log \left(c_3 + \frac{P}{(\alpha_0 + \alpha_2)P + N_3} \right). \quad (32)$$

Thus, *Theorem 8* is proved.

C. The Gap

We choose the parameter $\alpha_0 = \frac{N_2}{P}$, which is suboptimal but good enough to achieve a constant gap. This choice of parameter, inspired by [1], ensures making efficient use of signal scale difference between N_1 and N_2 at receiver 1, while keeping the interference of \mathbf{x}_{10} at the noise level N_2 at receiver 2. By substitution, we get

$$T_1 = \frac{1}{2} \log \left(c_{11} + \frac{P - N_2}{\alpha_2 P + 2N_2} \right) + \frac{1}{2} \log \left(1 + \frac{N_2}{N_1} \right) \quad (33)$$

$$T_2 = \frac{1}{2} \log \left(1 + \frac{\alpha_2 P}{2N_2} \right) \quad (34)$$

$$T_3 \geq \frac{1}{2} \log \left(c_3 + \frac{P}{\alpha_2 P + N_2 + N_3} \right). \quad (35)$$

Since $\alpha_0 = \frac{N_2}{P} \in [0, \frac{1}{3}]$, it follows that $c_{11} = \frac{1 - N_2/P}{2 - N_2/P} \geq \frac{2}{5}$, and $c_3 = \frac{1}{2 - N_2/P} \geq \frac{1}{2}$.

Starting from \mathcal{R}_o from Table II, we can express the two-dimensional outer bound region at R_2 as

$$R_1 \leq \min \left\{ \frac{1}{2} \log \left(1 + \frac{2P}{N_1} \right) - R_2, C_1 \right\}$$

$$\leq \min \left\{ \frac{1}{2} \log \left(\frac{P}{N_1} \cdot \frac{7}{3} \right) - R_2, \frac{1}{2} \log \left(\frac{P}{N_1} \cdot \frac{4}{3} \right) \right\}$$

$$R_3 \leq \min \left\{ \frac{1}{2} \log \left(1 + \frac{2P}{N_2} \right) - R_2, C_3 \right\}$$

$$\leq \min \left\{ \frac{1}{2} \log \left(\frac{P}{N_2} \cdot \frac{7}{3} \right) - R_2, \frac{1}{2} \log \left(\frac{P}{N_3} \cdot \frac{4}{3} \right) \right\}.$$

Depending on the bottleneck of $\min\{\cdot, \cdot\}$ expressions, there are three cases:

- $R_2 \leq \frac{1}{2} \log \left(\frac{7}{4} \right)$
- $\frac{1}{2} \log \left(\frac{7}{4} \right) \leq R_2 \leq \frac{1}{2} \log \left(\frac{N_3}{N_2} \cdot \frac{7}{4} \right)$
- $R_2 \geq \frac{1}{2} \log \left(\frac{N_3}{N_2} \cdot \frac{7}{4} \right)$.

At $R_2 = \frac{1}{2} \log \left(\frac{\alpha_2 P}{N_2} \cdot \frac{7}{4} \right)$, the outer bound region is

$$R_1 \leq \min \left\{ \frac{1}{2} \log \left(\frac{P}{\alpha_2 P} \cdot \frac{N_2}{N_1} \cdot \frac{4}{3} \right), \frac{1}{2} \log \left(\frac{P}{N_1} \cdot \frac{4}{3} \right) \right\}$$

$$R_3 \leq \min \left\{ \frac{1}{2} \log \left(\frac{P}{\alpha_2 P} \cdot \frac{4}{3} \right), \frac{1}{2} \log \left(\frac{P}{N_3} \cdot \frac{4}{3} \right) \right\}.$$

Depending on the bottleneck of $\min\{\cdot, \cdot\}$ expressions, we consider the following three cases:

- $\alpha_2 P \geq N_3$
- $N_2 \leq \alpha_2 P \leq N_3$
- $\alpha_2 P \leq N_2$.

Case i) $\alpha_2 P \geq N_3$: The outer bound region at $R_2 = \frac{1}{2} \log \left(\frac{\alpha_2 P}{N_2} \cdot \frac{7}{4} \right)$ is

$$R_1 \leq \frac{1}{2} \log \left(\frac{P}{\alpha_2 P} \cdot \frac{N_2}{N_1} \cdot \frac{4}{3} \right), R_3 \leq \frac{1}{2} \log \left(\frac{P}{\alpha_2 P} \cdot \frac{4}{3} \right). \quad (36)$$

For comparison, let us take a look at the achievable rate region. The first term of T_1 is lower bounded by

$$T_{11}'' = \frac{1}{2} \log \left(c_{11} + \frac{P - N_2}{\alpha_2 P + 2N_2} \right) \quad (37)$$

$$\geq \frac{1}{2} \log \left(\frac{2}{5} + \frac{P - \alpha_2 P}{3\alpha_2 P} \right) \quad (38)$$

$$> \frac{1}{2} \log \left(\frac{P}{3\alpha_2 P} \right). \quad (39)$$

We get the lower bounds:

$$T_1 = T_{11}'' + T_{10} \quad (40)$$

$$> \frac{1}{2} \log \left(\frac{P}{3\alpha_2 P} \right) + \frac{1}{2} \log \left(1 + \frac{N_2}{N_1} \right) \quad (41)$$

$$> \frac{1}{2} \log \left(\frac{P}{3\alpha_2 P} \cdot \frac{N_2}{N_1} \right) \quad (42)$$

$$T_3 \geq \frac{1}{2} \log \left(\frac{1}{2} + \frac{P}{\alpha_2 P + N_2 + N_3} \right) \quad (43)$$

$$> \frac{1}{2} \log \left(\frac{P}{3\alpha_2 P} \right). \quad (44)$$

For fixed α_2 and $R_2 = \frac{1}{2} \log \left(\frac{\alpha_2 P}{2N_2} \right)$, the two-dimensional achievable rate region is given by

$$R_1 \leq \frac{1}{2} \log \left(\frac{P}{3\alpha_2 P} \cdot \frac{N_2}{N_1} \right), \quad R_3 \leq \frac{1}{2} \log \left(\frac{P}{3\alpha_2 P} \right). \quad (45)$$

Case ii) $N_2 \leq \alpha_2 P \leq N_3$: The outer bound region at $R_2 = \frac{1}{2} \log \left(\frac{\alpha_2 P}{N_2} \cdot \frac{7}{4} \right)$ is

$$R_1 \leq \frac{1}{2} \log \left(\frac{P}{\alpha_2 P} \cdot \frac{N_2}{N_1} \cdot \frac{4}{3} \right), \quad R_3 \leq \frac{1}{2} \log \left(\frac{P}{N_3} \cdot \frac{4}{3} \right). \quad (46)$$

Now, let us take a look at the achievable rate region. We have the lower bounds:

$$T_1 > \frac{1}{2} \log \left(\frac{P}{3\alpha_2 P} \cdot \frac{N_2}{N_1} \right) \quad (47)$$

$$T_3 \geq \frac{1}{2} \log \left(\frac{1}{2} + \frac{P}{\alpha_2 P + N_2 + N_3} \right) \quad (48)$$

$$> \frac{1}{2} \log \left(\frac{P}{3N_3} \right). \quad (49)$$

For fixed α_2 and $R_2 = \frac{1}{2} \log \left(\frac{\alpha_2 P}{2N_2} \right)$, the two-dimensional achievable rate region is given by

$$R_1 \leq \frac{1}{2} \log \left(\frac{P}{3\alpha_2 P} \cdot \frac{N_2}{N_1} \right), \quad R_3 \leq \frac{1}{2} \log \left(\frac{P}{3N_3} \right). \quad (50)$$

Case iii) $\alpha_2 P \leq N_2$: The outer bound region at $R_2 = \frac{1}{2} \log \left(\frac{\alpha_2 P}{N_2} \cdot \frac{7}{4} \right)$ is

$$R_1 \leq \frac{1}{2} \log \left(\frac{P}{N_1} \cdot \frac{4}{3} \right), \quad R_3 \leq \frac{1}{2} \log \left(\frac{P}{N_3} \cdot \frac{4}{3} \right). \quad (51)$$

For this range of α_2 , the rate R_2 is small, i.e., $R_2 = \frac{1}{2} \log \left(\frac{\alpha_2 P}{N_2} \cdot \frac{7}{4} \right) \leq \frac{1}{2} \log \left(\frac{7}{4} \right) < \frac{1}{2}$, and R_1 and R_3 are close to single user capacities C_1 and C_3 , respectively.

Let us take a look at the achievable rate region. The first term of T_1 is lower bounded by

$$T_{11}'' = \frac{1}{2} \log \left(c_{11} + \frac{P - N_2}{\alpha_2 P + 2N_2} \right) \quad (52)$$

$$\geq \frac{1}{2} \log \left(\frac{2}{5} + \frac{P - N_2}{3N_2} \right) \quad (53)$$

$$> \frac{1}{2} \log \left(\frac{P}{3N_2} \right). \quad (54)$$

We get the lower bounds:

$$T_1 = T_{11}'' + T_{10} \quad (55)$$

$$> \frac{1}{2} \log \left(\frac{P}{3N_2} \right) + \frac{1}{2} \log \left(1 + \frac{N_2}{N_1} \right) \quad (56)$$

$$> \frac{1}{2} \log \left(\frac{P}{3N_1} \right) \quad (57)$$

$$T_3 \geq \frac{1}{2} \log \left(\frac{1}{2} + \frac{P}{\alpha_2 P + N_2 + N_3} \right) \quad (58)$$

$$> \frac{1}{2} \log \left(\frac{P}{3N_3} \right). \quad (59)$$

For fixed α_2 and $R_2 = \frac{1}{2} \log \left(\frac{\alpha_2 P}{2N_2} \right)$, the following two-dimensional rate region is achievable.

$$R_1 \leq \frac{1}{2} \log \left(\frac{P}{3N_1} \right), \quad R_3 \leq \frac{1}{2} \log \left(\frac{P}{3N_3} \right). \quad (60)$$

In all three cases above, by comparing the inner and outer bound regions, we can see that $\delta_1 \leq \frac{1}{2} \log \left(3 \cdot \frac{4}{3} \right) = 1$, $\delta_2 \leq \frac{1}{2} \log \left(2 \cdot \frac{7}{4} \right) = 0.91$ and $\delta_3 \leq \frac{1}{2} \log \left(3 \cdot \frac{4}{3} \right) = 1$. Therefore, we can conclude that the gap is to within one bit per message.

IV. INNER BOUND: CHANNEL TYPE 2

Theorem 9: Given $\alpha_1 \in [0, 1]$, the region \mathcal{R}_α is defined by

$$\begin{aligned} R_1 &\leq \frac{1}{2} \log \left(1 + \frac{\alpha_1 P}{N_1} \right) \\ R_2 &\leq \frac{1}{2} \log^+ \left(\frac{1}{2} + \frac{P}{\alpha_1 P + N_2} \right) \\ R_3 &\leq \frac{1}{2} \log^+ \left(\frac{1}{2} + \frac{P}{\alpha_1 P + N_3} \right), \end{aligned}$$

and $\mathcal{R} = \text{CONV} \left(\bigcup_{\alpha_1} \mathcal{R}_\alpha \right)$ is achievable.

A. Achievable Scheme

For this channel type, rate splitting is not necessary. Transmit signal \mathbf{x}_k is a coded signal of $M_k \in \{1, 2, \dots, 2^{nR_k}\}$, $k = 1, 2, 3$. In particular, \mathbf{x}_2 and \mathbf{x}_3 are lattice-coded signals using the same pair of coding and shaping lattices. As a result, the sum $\mathbf{x}_2 + \mathbf{x}_3$ is a dithered lattice codeword. The power allocation satisfies $\mathbb{E}[\|\mathbf{x}_1\|^2] = \alpha_1 nP$, $\mathbb{E}[\|\mathbf{x}_2\|^2] = nP$, and $\mathbb{E}[\|\mathbf{x}_3\|^2] = nP$. The received signals are

$$\mathbf{y}_1 = [\mathbf{x}_2 + \mathbf{x}_3] + \mathbf{x}_1 + \mathbf{z}_1$$

$$\mathbf{y}_2 = \mathbf{x}_2 + \mathbf{x}_1 + \mathbf{z}_2$$

$$\mathbf{y}_3 = \mathbf{x}_3 + \mathbf{x}_1 + \mathbf{z}_3.$$

The signal scale diagram at each receiver is shown in Fig. 3 (b). Decoding is performed in the following way.

- At receiver 1, $[\mathbf{x}_2 + \mathbf{x}_3]$ is first decoded while treating $\mathbf{x}_1 + \mathbf{z}_1$ as noise. Next, \mathbf{x}_1 is decoded from $\mathbf{y}_1 - [\mathbf{x}_2 + \mathbf{x}_3] = \mathbf{x}_1 + \mathbf{z}_1$. For reliable decoding, the code rates should satisfy

$$R_2 \leq T'_2 = \frac{1}{2} \log \left(\frac{1}{2} + \frac{P}{\alpha_1 P + N_1} \right) \quad (61)$$

$$R_3 \leq T'_3 = \frac{1}{2} \log \left(\frac{1}{2} + \frac{P}{\alpha_1 P + N_1} \right) \quad (62)$$

$$R_1 \leq T_1 = \frac{1}{2} \log \left(1 + \frac{\alpha_1 P}{N_1} \right). \quad (63)$$

- At receiver 2, \mathbf{x}_2 is decoded while treating $\mathbf{x}_1 + \mathbf{z}_2$ as noise. Similarly at receiver 3, \mathbf{x}_3 is decoded while treating $\mathbf{x}_1 + \mathbf{z}_3$ as noise. For reliable decoding, the code rates should satisfy

$$R_2 \leq T''_2 = \frac{1}{2} \log \left(1 + \frac{P}{\alpha_1 P + N_2} \right) \quad (64)$$

$$R_3 \leq T''_3 = \frac{1}{2} \log \left(1 + \frac{P}{\alpha_1 P + N_3} \right). \quad (65)$$

Putting together, we get

$$\begin{aligned} R_1 &\leq T_1 \\ R_2 &\leq T_2 = \min\{T'_2, T''_2\} \\ R_3 &\leq T_3 = \min\{T'_3, T''_3\} \end{aligned}$$

where

$$T_1 = \frac{1}{2} \log \left(1 + \frac{\alpha_1 P}{N_1} \right) \quad (66)$$

$$T_2 \geq \frac{1}{2} \log \left(\frac{1}{2} + \frac{P}{\alpha_1 P + N_2} \right) \quad (67)$$

$$\geq \frac{1}{2} \log \left(\frac{1}{2} + \frac{P}{2 \cdot \max\{\alpha_1 P, N_2\}} \right) \quad (68)$$

$$T_3 \geq \frac{1}{2} \log \left(\frac{1}{2} + \frac{P}{\alpha_1 P + N_3} \right) \quad (69)$$

$$\geq \frac{1}{2} \log \left(\frac{1}{2} + \frac{P}{2 \cdot \max\{\alpha_1 P, N_3\}} \right). \quad (70)$$

B. The Gap

Starting from \mathcal{R}_o from Table II, we can express the two-dimensional outer bound region at R_1 as

$$\begin{aligned} R_2 &\leq \min \left\{ \frac{1}{2} \log \left(1 + \frac{2P}{N_1} \right) - R_1, C_2 \right\} \\ &\leq \min \left\{ \frac{1}{2} \log \left(\frac{P}{N_1} \cdot \frac{7}{3} \right) - R_1, \frac{1}{2} \log \left(\frac{P}{N_2} \cdot \frac{4}{3} \right) \right\} \\ R_3 &\leq \min \left\{ \frac{1}{2} \log \left(1 + \frac{2P}{N_1} \right) - R_1, C_3 \right\} \\ &\leq \min \left\{ \frac{1}{2} \log \left(\frac{P}{N_1} \cdot \frac{7}{3} \right) - R_1, \frac{1}{2} \log \left(\frac{P}{N_3} \cdot \frac{4}{3} \right) \right\}. \end{aligned}$$

Depending on the bottleneck of $\min\{\cdot, \cdot\}$ expressions, there are three cases:

- $R_1 \leq \frac{1}{2} \log \left(\frac{N_2}{N_1} \cdot \frac{7}{4} \right)$
- $\frac{1}{2} \log \left(\frac{N_2}{N_1} \cdot \frac{7}{4} \right) \leq R_1 \leq \frac{1}{2} \log \left(\frac{N_3}{N_1} \cdot \frac{7}{4} \right)$

$$\bullet R_1 \geq \frac{1}{2} \log \left(\frac{N_3}{N_1} \cdot \frac{7}{4} \right).$$

At $R_1 = \frac{1}{2} \log \left(\frac{\alpha_1 P}{N_1} \cdot \frac{7}{4} \right)$, the region can be expressed as

$$\begin{aligned} R_2 &\leq \min \left\{ \frac{1}{2} \log \left(\frac{P}{\alpha_1 P} \cdot \frac{4}{3} \right), \frac{1}{2} \log \left(\frac{P}{N_2} \cdot \frac{4}{3} \right) \right\} \\ R_3 &\leq \min \left\{ \frac{1}{2} \log \left(\frac{P}{\alpha_1 P} \cdot \frac{4}{3} \right), \frac{1}{2} \log \left(\frac{P}{N_3} \cdot \frac{4}{3} \right) \right\}. \end{aligned}$$

Depending on the bottleneck of $\min\{\cdot, \cdot\}$ expressions, we consider the following three cases.

Case i) $\alpha_1 P \geq N_3$: The two-dimensional outer bound region at $R_1 = \frac{1}{2} \log \left(\frac{\alpha_1 P}{N_1} \cdot \frac{7}{4} \right)$ is

$$R_2 \leq \frac{1}{2} \log \left(\frac{P}{\alpha_1 P} \cdot \frac{4}{3} \right), \quad R_3 \leq \frac{1}{2} \log \left(\frac{P}{\alpha_1 P} \cdot \frac{4}{3} \right). \quad (71)$$

For fixed α_1 and $R_1 = \frac{1}{2} \log \left(\frac{\alpha_1 P}{N_1} \right)$, the following two-dimensional region is achievable.

$$R_2 \leq \frac{1}{2} \log \left(\frac{P}{2\alpha_1 P} \right), \quad R_3 \leq \frac{1}{2} \log \left(\frac{P}{2\alpha_1 P} \right). \quad (72)$$

Case ii) $N_2 \leq \alpha_1 P \leq N_3$: The two-dimensional outer bound region at $R_1 = \frac{1}{2} \log \left(\frac{\alpha_1 P}{N_1} \cdot \frac{7}{4} \right)$ is

$$R_2 \leq \frac{1}{2} \log \left(\frac{P}{\alpha_1 P} \cdot \frac{4}{3} \right), \quad R_3 \leq \frac{1}{2} \log \left(\frac{P}{N_3} \cdot \frac{4}{3} \right). \quad (73)$$

For fixed α_1 and $R_1 = \frac{1}{2} \log \left(\frac{\alpha_1 P}{N_1} \right)$, the following two-dimensional region is achievable.

$$R_2 \leq \frac{1}{2} \log \left(\frac{P}{2\alpha_1 P} \right), \quad R_3 \leq \frac{1}{2} \log \left(\frac{P}{2N_3} \right). \quad (74)$$

Case iii) $\alpha_1 P \leq N_2$: The two-dimensional outer bound region at $R_1 = \frac{1}{2} \log \left(\frac{\alpha_1 P}{N_1} \cdot \frac{7}{4} \right)$ is

$$R_2 \leq \frac{1}{2} \log \left(\frac{P}{N_2} \cdot \frac{4}{3} \right), \quad R_3 \leq \frac{1}{2} \log \left(\frac{P}{N_3} \cdot \frac{4}{3} \right). \quad (75)$$

For fixed α_1 and $R_1 = \frac{1}{2} \log \left(\frac{\alpha_1 P}{N_1} \right)$, the following two-dimensional region is achievable.

$$R_2 \leq \frac{1}{2} \log \left(\frac{P}{2N_2} \right), \quad R_3 \leq \frac{1}{2} \log \left(\frac{P}{2N_3} \right). \quad (76)$$

In all three cases above, by comparing the inner and outer bounds, we can see that $\delta_1 \leq \frac{1}{2} \log \left(\frac{7}{4} \right) < 0.41$, $\delta_2 \leq \frac{1}{2} \log \left(2 \cdot \frac{4}{3} \right) < 0.71$, and $\delta_3 \leq \frac{1}{2} \log \left(2 \cdot \frac{4}{3} \right) < 0.71$. We can conclude that the inner and outer bounds are to within one bit.

V. INNER BOUND: CHANNEL TYPE 3

Theorem 10: Given $\alpha \in [0, 1]$, the region \mathcal{R}_α is defined by

$$\begin{aligned} R_1 &\leq \frac{1}{2} \log \left(1 + \frac{\alpha P}{N_1} \right) \\ R_2 &\leq \frac{1}{2} \log \left(1 + \frac{\alpha P}{N_2} \right) \\ R_3 &\leq \frac{1}{2} \log \left(1 + \frac{P}{2\alpha P + N_3} \right), \end{aligned}$$

and $\mathcal{R} = \text{CONV}(\bigcup_\alpha \mathcal{R}_\alpha)$ is achievable.

A. Achievable Scheme

For this channel type, neither rate splitting nor aligned interference decoding is necessary. Transmit signal \mathbf{x}_k is a coded signal of $M_k \in \{1, 2, \dots, 2^{nR_k}\}$, $k = 1, 2, 3$. The power allocation satisfies $\mathbb{E}[\|\mathbf{x}_1\|^2] = \alpha nP$, $\mathbb{E}[\|\mathbf{x}_2\|^2] = \alpha nP$, and $\mathbb{E}[\|\mathbf{x}_3\|^2] = nP$. The received signals are

$$\begin{aligned} \mathbf{y}_1 &= \mathbf{x}_3 + \mathbf{x}_1 + \mathbf{z}_1 \\ \mathbf{y}_2 &= \mathbf{x}_3 + \mathbf{x}_2 + \mathbf{z}_2 \\ \mathbf{y}_3 &= \mathbf{x}_3 + \mathbf{x}_1 + \mathbf{x}_2 + \mathbf{z}_3. \end{aligned}$$

The signal scale diagram at each receiver is shown in Fig. 3 (c). Decoding is performed in the following way.

- At receiver 1, \mathbf{x}_3 is first decoded while treating $\mathbf{x}_1 + \mathbf{z}_1$ as noise. Next, \mathbf{x}_1 is decoded from $\mathbf{y}_1 - \mathbf{x}_3 = \mathbf{x}_1 + \mathbf{z}_1$. For reliable decoding, the code rates should satisfy

$$R_3 \leq T'_3 = \frac{1}{2} \log \left(1 + \frac{P}{\alpha P + N_1} \right) \quad (77)$$

$$R_1 \leq T_1 = \frac{1}{2} \log \left(1 + \frac{\alpha P}{N_1} \right). \quad (78)$$

- At receiver 2, \mathbf{x}_3 is first decoded while treating $\mathbf{x}_2 + \mathbf{z}_2$ as noise. Next, \mathbf{x}_2 is decoded from $\mathbf{y}_2 - \mathbf{x}_3 = \mathbf{x}_2 + \mathbf{z}_2$. For reliable decoding, the code rates should satisfy

$$R_3 \leq T''_3 = \frac{1}{2} \log \left(1 + \frac{P}{\alpha P + N_2} \right) \quad (79)$$

$$R_2 \leq T_2 = \frac{1}{2} \log \left(1 + \frac{\alpha P}{N_2} \right). \quad (80)$$

- At receiver 3, \mathbf{x}_3 is decoded while treating $\mathbf{x}_1 + \mathbf{x}_2 + \mathbf{z}_3$ as noise. For reliable decoding, the code rates should satisfy

$$R_3 \leq T'''_3 = \frac{1}{2} \log \left(1 + \frac{P}{2\alpha P + N_3} \right). \quad (81)$$

Putting together, we get

$$\begin{aligned} R_1 &\leq T_1 \\ R_2 &\leq T_2 \\ R_3 &\leq T_3 = \min\{T'_3, T''_3, T'''_3\} \end{aligned}$$

where

$$T_1 = \frac{1}{2} \log \left(1 + \frac{\alpha P}{N_1} \right) \quad (82)$$

$$T_2 = \frac{1}{2} \log \left(1 + \frac{\alpha P}{N_2} \right) \quad (83)$$

$$T_3 = \frac{1}{2} \log \left(1 + \frac{P}{2\alpha P + N_3} \right) \quad (84)$$

$$\geq \frac{1}{2} \log \left(1 + \frac{P}{3 \cdot \max\{\alpha P, N_3\}} \right). \quad (85)$$

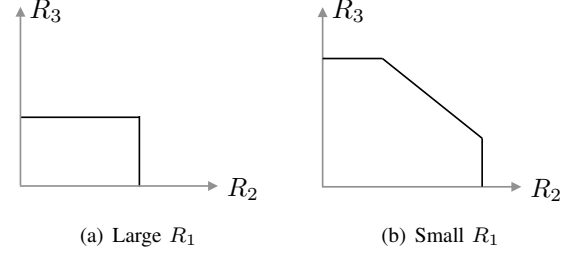


Fig. 4. The cross-section of the type 4 outer bound region at a relatively small or large R_1 .

B. The Gap

Starting from \mathcal{R}_o from Table II, we can express the two-dimensional outer bound region at R_3 as

$$\begin{aligned} R_1 &\leq \min \left\{ \frac{1}{2} \log \left(1 + \frac{2P}{N_1} \right) - R_3, C_1 \right\} \\ &\leq \min \left\{ \frac{1}{2} \log \left(\frac{P}{N_1} \cdot \frac{7}{3} \right) - R_3, \frac{1}{2} \log \left(\frac{P}{N_1} \cdot \frac{4}{3} \right) \right\} \\ R_2 &\leq \min \left\{ \frac{1}{2} \log \left(1 + \frac{2P}{N_2} \right) - R_3, C_2 \right\} \\ &\leq \min \left\{ \frac{1}{2} \log \left(\frac{P}{N_2} \cdot \frac{7}{3} \right) - R_3, \frac{1}{2} \log \left(\frac{P}{N_2} \cdot \frac{4}{3} \right) \right\}. \end{aligned}$$

Depending on the bottleneck of $\min\{\cdot, \cdot\}$ expressions, there are two cases: $R_3 \leq \frac{1}{2} \log \left(\frac{7}{4} \right)$ and $R_3 \geq \frac{1}{2} \log \left(\frac{7}{4} \right)$. We assume that $R_3 \geq \frac{1}{2} \log \left(\frac{7}{4} \right)$, equivalently $\alpha \leq \frac{4}{7}$. We also assume that $R_3 \leq \frac{1}{2} \log \left(\frac{P}{N_3} \right)$, equivalently $\alpha P \geq N_3$. The other cases are trivial.

The two-dimensional outer bound region at $R_3 = \frac{1}{2} \log \left(\frac{P}{\alpha P} \right)$ is

$$\begin{aligned} R_1 &\leq \min \left\{ \frac{1}{2} \log \left(\frac{\alpha P}{N_1} \cdot \frac{7}{3} \right), \frac{1}{2} \log \left(\frac{P}{N_1} \cdot \frac{4}{3} \right) \right\} \\ R_2 &\leq \min \left\{ \frac{1}{2} \log \left(\frac{\alpha P}{N_2} \cdot \frac{7}{3} \right), \frac{1}{2} \log \left(\frac{P}{N_2} \cdot \frac{4}{3} \right) \right\}. \end{aligned}$$

For $\alpha \leq \frac{4}{7}$, the two-dimensional outer bound region is

$$R_1 \leq \frac{1}{2} \log \left(\frac{\alpha P}{N_1} \cdot \frac{7}{3} \right), \quad R_2 \leq \frac{1}{2} \log \left(\frac{\alpha P}{N_2} \cdot \frac{7}{3} \right). \quad (86)$$

For $\alpha P \geq N_3$, the two-dimensional achievable rate region at $R_3 = \frac{1}{2} \log \left(\frac{P}{3\alpha P} \right)$ is

$$R_1 \leq \frac{1}{2} \log \left(\frac{\alpha P}{N_1} \right), \quad R_2 \leq \frac{1}{2} \log \left(\frac{\alpha P}{N_2} \right). \quad (87)$$

By comparing the inner and outer bounds, we can see that $\delta_1 \leq \frac{1}{2} \log \left(\frac{7}{3} \right) < 0.62$, $\delta_2 \leq \frac{1}{2} \log \left(\frac{7}{3} \right) < 0.62$, and $\delta_3 \leq \frac{1}{2} \log (3) < 0.8$. We can conclude that the inner and outer bounds are to within one bit.

VI. INNER BOUND: CHANNEL TYPE 4

The relaxed outer bound region \mathcal{R}'_o given by

$$\begin{aligned} R_k &\leq \frac{1}{2} \log \left(\frac{P}{N_k} \right) + \frac{1}{2} \log \left(\frac{4}{3} \right), \quad k = 1, 2, 3 \\ R_1 + R_2 &\leq \frac{1}{2} \log \left(\frac{P}{N_1} \right) + \frac{1}{2} \log \left(\frac{7}{3} \right) \\ R_1 + R_3 &\leq \frac{1}{2} \log \left(\frac{P}{N_1} \right) + \frac{1}{2} \log \left(\frac{7}{3} \right) \\ R_2 + R_3 &\leq \frac{1}{2} \log \left(\frac{P}{N_2} \right) + \frac{1}{2} \log \left(\frac{7}{3} \right). \end{aligned}$$

The cross-sectional region at a given R_1 is described by

$$\begin{aligned} R_2 &\leq \min \left\{ \frac{1}{2} \log \left(\frac{P}{N_1} \cdot \frac{7}{3} \right) - R_1, \frac{1}{2} \log \left(\frac{P}{N_2} \cdot \frac{4}{3} \right) \right\} \\ R_3 &\leq \min \left\{ \frac{1}{2} \log \left(\frac{P}{N_1} \cdot \frac{7}{3} \right) - R_1, \frac{1}{2} \log \left(\frac{P}{N_3} \cdot \frac{4}{3} \right) \right\} \\ R_2 + R_3 &\leq \frac{1}{2} \log \left(\frac{P}{N_2} \cdot \frac{7}{3} \right). \end{aligned}$$

Depending on the bottleneck of $\min\{\cdot, \cdot\}$ expressions, there are three cases:

- $R_1 \leq \frac{1}{2} \log \left(\frac{N_2}{N_1} \cdot \frac{7}{4} \right)$
- $\frac{1}{2} \log \left(\frac{N_2}{N_1} \cdot \frac{7}{4} \right) \leq R_1 \leq \frac{1}{2} \log \left(\frac{N_3}{N_1} \cdot \frac{7}{4} \right)$
- $R_1 \geq \frac{1}{2} \log \left(\frac{N_3}{N_1} \cdot \frac{7}{4} \right)$.

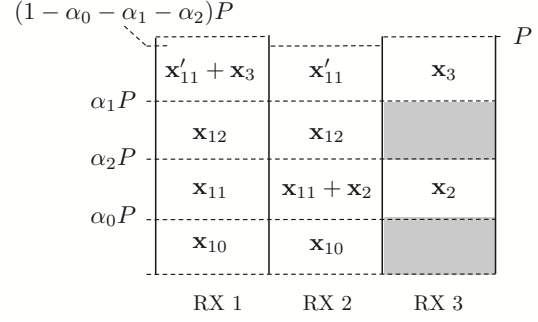
In this section, we focus on the third case. The other cases can be proved similarly. If the sum of the righthand sides of R_2 and R_3 bounds is smaller than the righthand side of $R_2 + R_3$ bound, i.e.,

$$\log \left(\frac{P}{N_1} \cdot \frac{7}{3} \right) - 2R_1 \leq \frac{1}{2} \log \left(\frac{P}{N_2} \cdot \frac{7}{3} \right), \quad (88)$$

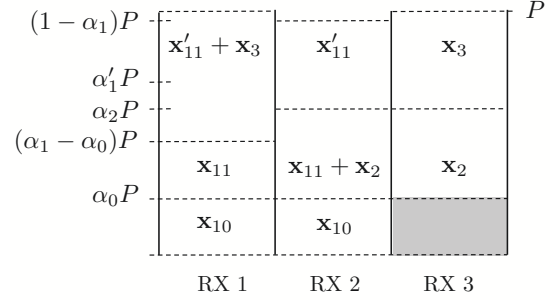
then the $R_2 + R_3$ bound is not active at the R_1 . This condition can be expressed as a threshold on R_1 given by

$$\begin{aligned} R_1 &> R_{1,th} = \frac{1}{2} \log \left(\frac{P}{N_1} \cdot \frac{7}{3} \right) - \frac{1}{4} \log \left(\frac{P}{N_2} \cdot \frac{7}{3} \right) \\ &= \frac{1}{4} \log \left(\frac{P}{N_1} \cdot \frac{7}{3} \right) + \frac{1}{4} \log \left(\frac{N_2}{N_1} \right). \end{aligned} \quad (89)$$

For this relatively large R_1 , the cross-sectional region is a rectangle as described in Fig. 4 (a). In contrast, for a relatively small R_1 , when the threshold condition does not hold, the cross-sectional region is a MAC-like region as described in Fig. 4 (b). In the rest of the section, we present achievable schemes for each case.



(a) Channel type 4: relatively large R_1



(b) Channel type 4: relatively small R_1

Fig. 5. Signal scale diagram.

A. Achievable Scheme for Relatively Large R_1

Theorem 11: Given $\alpha = (\alpha_0, \alpha_1, \alpha_2) \in [0, 1]^3$, the region \mathcal{R}_α is defined by

$$\begin{aligned} R_1 &\leq \min \left\{ \frac{1}{2} \log^+ \left(c_{11} + \frac{(1 - \alpha_0 - \alpha_1 - \alpha_2)P}{(\alpha_0 + \alpha_1 + 2\alpha_2)P + N_2} \right), \right. \\ &\quad \left. \frac{1}{2} \log \left(1 + \frac{\alpha_2 P}{\alpha_0 P + N_1} \right) \right\} \\ &\quad + \frac{1}{2} \log \left(1 + \frac{\alpha_1 P}{(\alpha_0 + \alpha_2)P + N_2} \right) \\ &\quad + \frac{1}{2} \log \left(1 + \frac{\alpha_0 P}{N_1} \right) \\ R_2 &\leq \frac{1}{2} \log \left(1 + \frac{\alpha_2 P}{\alpha_0 P + N_2} \right) \\ R_3 &\leq \frac{1}{2} \log^+ \left(c_3 + \frac{P}{(\alpha_0 + \alpha_1 + \alpha_2)P + N_3} \right) \end{aligned}$$

where $c_{11} = \frac{1 - \alpha_0 - \alpha_1 - \alpha_2}{2 - \alpha_0 - \alpha_1 - \alpha_2}$ and $c_3 = \frac{1}{2 - \alpha_0 - \alpha_1 - \alpha_2}$, and $\mathcal{R} = \text{CONV}(\bigcup_\alpha \mathcal{R}_\alpha)$ is achievable.

We present an achievable scheme for the case of $R_1 > R_{1,th}$. Message $M_1 \in \{1, 2, \dots, 2^{nR_1}\}$ is split into three parts: $M_{10} \in \{1, 2, \dots, 2^{nR_{10}}\}$, $M_{11} \in \{1, 2, \dots, 2^{nR_{11}}\}$ and $M_{12} \in \{1, 2, \dots, 2^{nR_{12}}\}$, so $R_1 = R_{10} + R_{11} + R_{12}$. We generate the signals in the following way: \mathbf{x}_{11} and \mathbf{x}'_{11} are differently coded signals of M_{11} , and \mathbf{x}_{10} and \mathbf{x}_{12} are coded signal of M_{10} and M_{12} , respectively. The transmit signal is the sum

$$\mathbf{x}_1 = \mathbf{x}_{10} + \mathbf{x}_{11} + \mathbf{x}_{12} + \mathbf{x}'_{11}.$$

The power allocation satisfies $\mathbb{E}[\|\mathbf{x}_{10}\|^2] = \alpha_0 nP$, $\mathbb{E}[\|\mathbf{x}_{11}\|^2] = \alpha_2 nP$, $\mathbb{E}[\|\mathbf{x}_{12}\|^2] = \alpha_1 nP$, and $\mathbb{E}[\|\mathbf{x}'_{11}\|^2] = (1 - \alpha_0 - \alpha_1 - \alpha_2)nP$.

The transmit signals \mathbf{x}_2 and \mathbf{x}_3 are coded signals of the messages $M_2 \in \{1, 2, \dots, 2^{nR_2}\}$ and $M_3 \in \{1, 2, \dots, 2^{nR_3}\}$, satisfying $\mathbb{E}[\|\mathbf{x}_2\|^2] = \alpha_2 nP$ and $\mathbb{E}[\|\mathbf{x}_3\|^2] = nP$.

The signals \mathbf{x}'_{11} and \mathbf{x}_3 are lattice-coded signals using the same coding lattice but different shaping lattices. As a result, the sum $\mathbf{x}'_{11} + \mathbf{x}_3$ is a dithered lattice codeword.

The received signals are

$$\begin{aligned} \mathbf{y}_1 &= [\mathbf{x}'_{11} + \mathbf{x}_3] + \mathbf{x}_{12} + \mathbf{x}_{11} + \mathbf{x}_{10} + \mathbf{z}_1 \\ \mathbf{y}_2 &= \mathbf{x}'_{11} + \mathbf{x}_{12} + \mathbf{x}_{11} + \mathbf{x}_2 + \mathbf{x}_{10} + \mathbf{z}_2 \\ \mathbf{y}_3 &= \mathbf{x}_3 + \mathbf{x}_2 + \mathbf{z}_3. \end{aligned}$$

The signal scale diagram at each receiver is shown in Fig. 5 (a). Decoding is performed in the following way.

- At receiver 1, $[\mathbf{x}'_{11} + \mathbf{x}_3]$ is first decoded while treating other signals as noise and removed from \mathbf{y}_1 . Next, \mathbf{x}_{12} , \mathbf{x}_{11} , and \mathbf{x}_{10} are decoded successively. For reliable decoding, the code rates should satisfy

$$\begin{aligned} R_{11} &\leq T'_{11} = \frac{1}{2} \log \left(c_{11} + \frac{(1 - \alpha_0 - \alpha_1 - \alpha_2)P}{(\alpha_0 + \alpha_1 + \alpha_2)P + N_1} \right) \\ R_3 &\leq T'_3 = \frac{1}{2} \log \left(c_3 + \frac{P}{(\alpha_0 + \alpha_1 + \alpha_2)P + N_1} \right) \\ R_{12} &\leq T'_{12} = \frac{1}{2} \log \left(1 + \frac{\alpha_1 P}{(\alpha_0 + \alpha_2)P + N_1} \right) \\ R_{11} &\leq T''_{11} = \frac{1}{2} \log \left(1 + \frac{\alpha_2 P}{\alpha_0 P + N_1} \right) \\ R_{10} &\leq T_{10} = \frac{1}{2} \log \left(1 + \frac{\alpha_0 P}{N_1} \right) \end{aligned}$$

where $c_{11} = \frac{(1 - \alpha_0 - \alpha_1 - \alpha_2)P}{(1 - \alpha_0 - \alpha_1 - \alpha_2)P + P} = \frac{1 - \alpha_0 - \alpha_1 - \alpha_2}{2 - \alpha_0 - \alpha_1 - \alpha_2}$ and $c_3 = \frac{P}{(1 - \alpha_0 - \alpha_1 - \alpha_2)P + P} = \frac{1}{2 - \alpha_0 - \alpha_1 - \alpha_2}$. Note that $0 \leq c_{11} \leq \frac{1}{2}$, $c_{11} + c_3 = 1$, and $\frac{1}{2} \leq c_3 \leq 1$.

- At receiver 2, \mathbf{x}'_{11} is first decoded while treating other signals as noise. Having successfully recovered M_{11} , receiver 2 can generate \mathbf{x}_{11} and \mathbf{x}'_{11} , and cancel them from \mathbf{y}_2 . Next, \mathbf{x}_{12} is decoded from $\mathbf{x}_{12} + \mathbf{x}_2 + \mathbf{x}_{10} + \mathbf{z}_2$. Finally, \mathbf{x}_2 is decoded from $\mathbf{x}_2 + \mathbf{x}_{10} + \mathbf{z}_2$. For reliable decoding, the code rates should satisfy

$$\begin{aligned} R_{11} &\leq T'''_{11} = \frac{1}{2} \log \left(1 + \frac{(1 - \alpha_0 - \alpha_1 - \alpha_2)P}{(\alpha_0 + \alpha_1 + 2\alpha_2)P + N_2} \right) \\ R_{12} &\leq T''_{12} = \frac{1}{2} \log \left(1 + \frac{\alpha_1 P}{(\alpha_0 + \alpha_2)P + N_2} \right) \\ R_2 &\leq T_2 = \frac{1}{2} \log \left(1 + \frac{\alpha_2 P}{\alpha_0 P + N_2} \right). \end{aligned}$$

- At receiver 3, \mathbf{x}_3 is decoded while treating $\mathbf{x}_2 + \mathbf{z}_3$ as noise. Reliable decoding is possible if

$$R_3 \leq T'''_3 = \frac{1}{2} \log \left(1 + \frac{P}{\alpha_2 P + N_3} \right). \quad (90)$$

Putting together, we can see that given $\alpha_0, \alpha_1, \alpha_2 \in [0, 1]$, the following rate region is achievable.

$$\begin{aligned} R_1 &\leq T_1 = \min\{T'_{11}, T''_{11}, T'''_{11}\} + \min\{T'_{12}, T''_{12}\} + T_{10} \\ R_2 &\leq T_2 \\ R_3 &\leq T_3 = \min\{T'_3, T'''_3\} \end{aligned}$$

where

$$\begin{aligned} T_1 &= \min\{T'_{11}, T''_{11}, T'''_{11}\} + \min\{T'_{12}, T''_{12}\} + T_{10} \\ &= \min\{\min\{T'_{11}, T''_{11}\}, T'''_{11}\} + T'_{12} + T_{10} \\ &\geq \min \left\{ \frac{1}{2} \log \left(c_{11} + \frac{(1 - \alpha_0 - \alpha_1 - \alpha_2)P}{(\alpha_0 + \alpha_1 + 2\alpha_2)P + N_2} \right), \right. \\ &\quad \left. \frac{1}{2} \log \left(1 + \frac{\alpha_2 P}{\alpha_0 P + N_1} \right) \right\} \\ &\quad + \frac{1}{2} \log \left(1 + \frac{\alpha_1 P}{(\alpha_0 + \alpha_2)P + N_2} \right) \\ &\quad + \frac{1}{2} \log \left(1 + \frac{\alpha_0 P}{N_1} \right) \\ T_2 &= \frac{1}{2} \log \left(1 + \frac{\alpha_2 P}{\alpha_0 P + N_2} \right) \\ T_3 &\geq \frac{1}{2} \log \left(c_3 + \frac{P}{(\alpha_0 + \alpha_1 + \alpha_2)P + N_3} \right). \end{aligned}$$

B. The Gap for Relatively Large R_1

We choose α_0 , α_1 and α_2 such that $\alpha_1 \leq \frac{3}{8}$, that $\alpha_1 \geq 3(\alpha_0 + \alpha_2)$, that $\alpha_2 P \geq 3N_3$, and that $\alpha_0 P = N_2$. It follows that $\alpha_0 + \alpha_1 + \alpha_2 \leq \frac{4}{3}\alpha_1 \leq \frac{1}{2}$, that $c_{11} \geq \frac{1}{3}$, and that $(\alpha_0 + \alpha_1 + 2\alpha_2)P + N_2 = 2(\alpha_0 + \alpha_2)P + \alpha_1 P \leq \frac{5}{3}\alpha_1 P$. We get the lower bounds for each term of T_1 expression above.

$$\begin{aligned} &\min\{T'_{11}, T'''_{11}\} \\ &\geq \frac{1}{2} \log \left(c_{11} + \frac{(1 - \alpha_0 - \alpha_1 - \alpha_2)P}{(\alpha_0 + \alpha_1 + 2\alpha_2)P + N_2} \right) \\ &\geq \frac{1}{2} \log \left(\frac{1}{3} + \frac{(1 - (4/3)\alpha_1)P}{(5/3)\alpha_1 P} \right) \\ &= \frac{1}{2} \log \left(\frac{P}{(5/3)\alpha_1 P} - \frac{7}{15} \right) \\ &= \frac{1}{2} \log \left(\frac{P}{(5/3)\alpha_1 P} \right) + \frac{1}{2} \log \left(1 - \frac{7}{15} \cdot \frac{5}{3}\alpha_1 \right) \\ &\geq \frac{1}{2} \log \left(\frac{P}{(5/3)\alpha_1 P} \right) + \frac{1}{2} \log \left(\frac{17}{24} \right) \\ &\geq \frac{1}{2} \log \left(\frac{P}{\alpha_1 P} \cdot \frac{17}{40} \right) \end{aligned}$$

and

$$T''_{11} = \frac{1}{2} \log \left(1 + \frac{\alpha_2 P}{\alpha_0 P + N_1} \right) \quad (91)$$

$$= \frac{1}{2} \log \left(\frac{(\alpha_0 + \alpha_2)P + N_1}{\alpha_0 P + N_1} \right) \quad (92)$$

$$\geq \frac{1}{2} \log \left(\frac{(\alpha_0 + \alpha_2)P}{\alpha_0 P + N_2} \right) \quad (93)$$

$$= \frac{1}{2} \log \left(\frac{(\alpha_0 + \alpha_2)P}{2N_2} \right). \quad (94)$$

Since $(\alpha_0 + \alpha_2)P \geq N_2 + 3N_3 \geq 4N_2$,

$$T_{12}'' = \frac{1}{2} \log \left(1 + \frac{\alpha_1 P}{(\alpha_0 + \alpha_2)P + N_2} \right) \quad (95)$$

$$\geq \frac{1}{2} \log \left(\frac{\alpha_1 P}{(5/4)(\alpha_0 + \alpha_2)P} \right). \quad (96)$$

Putting together,

$$\begin{aligned} T_1 &\geq \min \left\{ \frac{1}{2} \log \left(\frac{P}{\alpha_1 P} \cdot \frac{17}{40} \right), \frac{1}{2} \log \left(\frac{(\alpha_0 + \alpha_2)P}{2N_2} \right) \right\} \\ &\quad + \frac{1}{2} \log \left(\frac{\alpha_1 P}{(5/4)(\alpha_0 + \alpha_2)P} \right) + \frac{1}{2} \log \left(\frac{N_2}{N_1} \right) \\ &= \min \left\{ \frac{1}{2} \log \left(\frac{P}{(\alpha_0 + \alpha_2)P} \cdot \frac{N_2}{N_1} \cdot \frac{17}{40} \cdot \frac{4}{5} \right), \right. \\ &\quad \left. \frac{1}{2} \log \left(\frac{\alpha_1 P}{N_1} \cdot \frac{1}{2} \cdot \frac{4}{5} \right) \right\} \\ &= \min \left\{ \frac{1}{2} \log \left(\frac{P}{(\alpha_0 + \alpha_2)P} \cdot \frac{N_2}{N_1} \cdot \frac{17}{50} \right), \right. \\ &\quad \left. \frac{1}{2} \log \left(\frac{\alpha_1 P}{N_1} \cdot \frac{2}{5} \right) \right\}. \end{aligned}$$

Given α_1 , we choose α_2 that satisfies $\frac{1}{2} \log \left(\frac{P}{\alpha_1 P} \cdot \frac{17}{40} \right) = \frac{1}{2} \log \left(\frac{(\alpha_0 + \alpha_2)P}{2N_2} \right)$. As a result, we can write $T_1 \geq \frac{1}{2} \log \left(\frac{\alpha_1 P}{N_1} \cdot \frac{2}{5} \right)$, and also

$$T_2 = \frac{1}{2} \log \left(1 + \frac{\alpha_2 P}{\alpha_0 P + N_2} \right) \quad (97)$$

$$\geq \frac{1}{2} \log \left(\frac{(\alpha_0 + \alpha_2)P}{2N_2} \right) \quad (98)$$

$$= \frac{1}{2} \log \left(\frac{P}{\alpha_1 P} \cdot \frac{17}{40} \right). \quad (99)$$

Since $N_3 \leq \frac{1}{3}\alpha_2 P \leq \frac{1}{3}(\alpha_0 + \alpha_2)P \leq \frac{1}{9}\alpha_1 P$,

$$\begin{aligned} T_3 &\geq \frac{1}{2} \log \left(c_3 + \frac{P}{(\alpha_0 + \alpha_1 + \alpha_2)P + N_3} \right) \\ &\geq \frac{1}{2} \log \left(\frac{1}{2} + \frac{P}{(4/3)\alpha_1 P + (1/9)\alpha_1 P} \right) \\ &\geq \frac{1}{2} \log \left(\frac{P}{(13/9)\alpha_1 P} \right). \end{aligned}$$

The following rate region is achievable.

$$R_1 \leq \frac{1}{2} \log \left(\frac{\alpha_1 P}{N_1} \cdot \frac{2}{5} \right) \quad (100)$$

$$R_2 \leq \frac{1}{2} \log \left(\frac{P}{\alpha_1 P} \cdot \frac{17}{40} \right) \quad (101)$$

$$R_3 \leq \frac{1}{2} \log \left(\frac{P}{\alpha_1 P} \cdot \frac{9}{13} \right). \quad (102)$$

For fixed α_1 and $R_1 = \frac{1}{2} \log \left(\frac{\alpha_1 P}{N_1} \cdot \frac{2}{5} \right)$, the two-dimensional rate region, given by

$$R_2 \leq \frac{1}{2} \log \left(\frac{P}{\alpha_1 P} \cdot \frac{17}{40} \right), \quad R_3 \leq \frac{1}{2} \log \left(\frac{P}{\alpha_1 P} \cdot \frac{9}{13} \right)$$

is achievable.

In comparison, the two-dimensional outer bound region at $R_1 = \frac{1}{2} \log \left(\frac{\alpha_1 P}{N_1} \cdot \frac{2}{5} \right) + 1$, given by

$$\begin{aligned} R_2 &\leq \frac{1}{2} \log \left(\frac{P}{N_1} \cdot \frac{7}{3} \right) - \frac{1}{2} \log \left(\frac{\alpha_1 P}{N_1} \cdot \frac{2}{5} \right) - 1 \\ &= \frac{1}{2} \log \left(\frac{P}{\alpha_1 P} \right) + \frac{1}{2} \log \left(\frac{7}{3} \cdot \frac{5}{2} \cdot \frac{1}{4} \right) \\ R_3 &\leq \frac{1}{2} \log \left(\frac{P}{N_1} \cdot \frac{7}{3} \right) - \frac{1}{2} \log \left(\frac{\alpha_1 P}{N_1} \cdot \frac{2}{5} \right) - 1 \\ &= \frac{1}{2} \log \left(\frac{P}{\alpha_1 P} \right) + \frac{1}{2} \log \left(\frac{7}{3} \cdot \frac{5}{2} \cdot \frac{1}{4} \right). \end{aligned}$$

As discussed above, the sum-rate bound on $R_2 + R_3$ is loose for R_1 larger than the threshold, so the rate region is a rectangle. By comparing the inner and outer bound rate regions, we can see that $\delta_2 < \frac{1}{2} \log \left(\frac{40}{17} \cdot \frac{7}{3} \cdot \frac{5}{2} \cdot \frac{1}{4} \right) < 0.89$ and $\delta_3 < \frac{1}{2} \log \left(\frac{13}{9} \cdot \frac{7}{3} \cdot \frac{5}{2} \cdot \frac{1}{4} \right) < 0.54$. Therefore, we can conclude that the gap is to within one bit per message.

C. Achievable Scheme for Relatively Small R_1

Theorem 12: Given $\alpha = (\alpha_0, \alpha_1, \alpha_2) \in [0, 1]^3$, the region \mathcal{R}_α is defined by

$$\begin{aligned} R_1 &\leq \min \left\{ \frac{1}{2} \log^+ \left(c_{11} + \frac{(1 - \alpha_1)P}{(\alpha_1 + \alpha_2)P + N_2} \right), \right. \\ &\quad \left. \frac{1}{2} \log \left(1 + \frac{(\alpha_1 - \alpha_0)P}{\alpha_0 P + N_1} \right) \right\} + \frac{1}{2} \log \left(1 + \frac{\alpha_0 P}{N_1} \right) \\ R_2 &\leq \frac{1}{2} \log \left(1 + \frac{\alpha_2 P}{\alpha_0 P + N_2} \right) \\ R_3 &\leq \frac{1}{2} \log^+ \left(c_3 + \frac{P}{\max\{\alpha_1, \alpha_2\}P + N_3} \right) \end{aligned}$$

where $c_{11} = \frac{1 - \alpha_1}{2 - \alpha_1}$ and $c_3 = \frac{1}{2 - \alpha_1}$, and $\mathcal{R} = \text{CONV}(\bigcup_\alpha \mathcal{R}_\alpha)$ is achievable.

For the case of $R_1 < R_{1,th}$, we present the following achievable scheme. At transmitter 1, we split M_1 into M_{10} and M_{11} , so $R_1 = R_{10} + R_{11}$. The transmit signal is the sum

$$\mathbf{x}_1 = \mathbf{x}_{10} + \mathbf{x}_{11} + \mathbf{x}'_{11}.$$

The power allocation satisfies $\mathbb{E}[\|\mathbf{x}_{10}\|^2] = \alpha_0 nP$, $\mathbb{E}[\|\mathbf{x}_{11}\|^2] = (\alpha_1 - \alpha_0)nP$, and $\mathbb{E}[\|\mathbf{x}'_{11}\|^2] = (1 - \alpha_1)nP$ at receiver 1, $\mathbb{E}[\|\mathbf{x}_2\|^2] = \alpha_2 nP$ at receiver 2, and $\mathbb{E}[\|\mathbf{x}_3\|^2] = nP$ at receiver 3.

The signals \mathbf{x}'_{11} and \mathbf{x}_3 are lattice codewords using the same coding lattice but different shaping lattices. As a result, the sum $\mathbf{x}'_{11} + \mathbf{x}_3$ is a lattice codeword.

The received signals are

$$\begin{aligned} \mathbf{y}_1 &= [\mathbf{x}'_{11} + \mathbf{x}_3] + \mathbf{x}_{11} + \mathbf{x}_{10} + \mathbf{z}_1 \\ \mathbf{y}_2 &= \mathbf{x}'_{11} + \mathbf{x}_{11} + \mathbf{x}_2 + \mathbf{x}_{10} + \mathbf{z}_2 \\ \mathbf{y}_3 &= \mathbf{x}_3 + \mathbf{x}_2 + \mathbf{z}_3. \end{aligned}$$

The signal scale diagram at each receiver is shown in Fig. 5 (b). Decoding is performed in the following way.

- At receiver 1, $[\mathbf{x}'_{11} + \mathbf{x}_3]$ is first decoded while treating other signals as noise and removed from \mathbf{y}_1 . Next,

\mathbf{x}_{11} and then \mathbf{x}_{10} is decoded successively. For reliable decoding, the code rates should satisfy

$$\begin{aligned} R_{11} &\leq T'_{11} = \frac{1}{2} \log \left(c_{11} + \frac{(1-\alpha_1)P}{\alpha_1 P + N_1} \right) \\ R_3 &\leq T'_3 = \frac{1}{2} \log \left(c_3 + \frac{P}{\alpha_1 P + N_1} \right) \\ R_{11} &\leq T''_{11} = \frac{1}{2} \log \left(1 + \frac{(\alpha_1 - \alpha_0)P}{\alpha_0 P + N_1} \right) \\ R_{10} &\leq T_{10} = \frac{1}{2} \log \left(1 + \frac{\alpha_0 P}{N_1} \right) \end{aligned}$$

where $c_{11} = \frac{(1-\alpha_1)P}{(1-\alpha_1)P+P} = \frac{1-\alpha_1}{2-\alpha_1}$ and $c_3 = \frac{P}{(1-\alpha_1)P+P} = \frac{1}{2-\alpha_1}$. Note that $0 \leq c_{11} \leq \frac{1}{2}$, $c_{11} + c_3 = 1$, and $\frac{1}{2} \leq c_3 \leq 1$.

- At receiver 2, \mathbf{x}'_{11} is first decoded while treating other signals as noise. Having successfully recovered M_{11} , receiver 1 can generate \mathbf{x}_{11} and \mathbf{x}'_{11} , and cancel them from \mathbf{y}_2 . Next, \mathbf{x}_2 is decoded from $\mathbf{x}_2 + \mathbf{x}_{10} + \mathbf{z}_2$. At receiver 2, \mathbf{x}_{10} is not decoded. For reliable decoding, the code rates should satisfy

$$\begin{aligned} R_{11} &\leq T'''_{11} = \frac{1}{2} \log \left(1 + \frac{(1-\alpha_1)P}{(\alpha_1 + \alpha_2)P + N_2} \right) \\ R_2 &\leq T_2 = \frac{1}{2} \log \left(1 + \frac{\alpha_2 P}{\alpha_0 P + N_2} \right). \end{aligned}$$

- At receiver 3, \mathbf{x}_3 is decoded while treating $\mathbf{x}_2 + \mathbf{z}_3$ as noise. Reliable decoding is possible if

$$R_3 \leq T'_3 = \frac{1}{2} \log \left(1 + \frac{P}{\alpha_2 P + N_3} \right). \quad (103)$$

and

Putting together, we can see that given $\alpha_0, \alpha_1, \alpha_2 \in [0, 1]$, the following rate region is achievable.

$$R_1 \leq T_1 = \min\{T'_{11}, T''_{11}, T'''_{11}\} + T_{10} \quad (104)$$

$$R_2 \leq T_2 \quad (105)$$

$$R_3 \leq T_3 = \min\{T'_3, T''_3\} \quad (106)$$

where

$$\begin{aligned} T_1 &= \min\{T'_{11}, T''_{11}, T'''_{11}\} + T_{10} \\ &= \min\{\min\{T'_{11}, T''_{11}\}, T'''_{11}\} + T_{10} \\ &\geq \min \left\{ \frac{1}{2} \log \left(c_{11} + \frac{(1-\alpha_1)P}{(\alpha_1 + \alpha_2)P + N_2} \right), \right. \\ &\quad \left. \frac{1}{2} \log \left(1 + \frac{(\alpha_1 - \alpha_0)P}{\alpha_0 P + N_1} \right) \right\} + \frac{1}{2} \log \left(1 + \frac{\alpha_0 P}{N_1} \right) \\ T_2 &= \frac{1}{2} \log \left(1 + \frac{\alpha_2 P}{\alpha_0 P + N_2} \right) \\ T_3 &\geq \frac{1}{2} \log \left(c_3 + \frac{P}{\max\{\alpha_1, \alpha_2\}P + N_3} \right). \end{aligned}$$

D. The Gap for Relatively Small R_1

We choose α_0, α_1 , and α_2 such that $\alpha_1 \leq \alpha_2 \leq \frac{1}{2}$, that $\alpha_1 P \geq 3N_2$, that $\alpha_2 P \geq 3N_3$, and that $\alpha_0 P = \frac{4}{5}N_2$. It

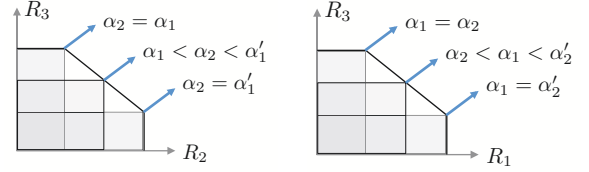


Fig. 6. MAC-like region.

follows that $c_{11} \geq \frac{1}{3}$ and that $(\alpha_1 + \alpha_2)P + N_2 \leq \frac{4}{3}\alpha_1 P + \alpha_2 P \leq \frac{7}{3}\alpha_2 P$.

$$\begin{aligned} &\min\{T'_{11}, T'''_{11}\} \\ &= \frac{1}{2} \log \left(c_{11} + \frac{(1-\alpha_1)P}{(\alpha_1 + \alpha_2)P + N_2} \right) \\ &\geq \frac{1}{2} \log \left(\frac{1}{3} + \frac{(1-\alpha_2)P}{(7/3)\alpha_2 P} \right) \\ &= \frac{1}{2} \log \left(\frac{P}{(7/3)\alpha_2 P} - \frac{2}{21} \right) \\ &= \frac{1}{2} \log \left(\frac{P}{(7/3)\alpha_2 P} \right) + \frac{1}{2} \log \left(1 - \frac{2}{21} \cdot \frac{7}{3} \alpha_2 \right) \\ &\geq \frac{1}{2} \log \left(\frac{P}{(7/3)\alpha_2 P} \right) + \frac{1}{2} \log \left(\frac{8}{9} \right) \\ &\geq \frac{1}{2} \log \left(\frac{P}{\alpha_2 P} \cdot \frac{8}{21} \right) \end{aligned}$$

$$T''_{11} = \frac{1}{2} \log \left(1 + \frac{(\alpha_1 - \alpha_0)P}{\alpha_0 P + N_1} \right) \quad (107)$$

$$= \frac{1}{2} \log \left(\frac{\alpha_1 P + N_1}{\alpha_0 P + N_1} \right) \quad (108)$$

$$\geq \frac{1}{2} \log \left(\frac{\alpha_1 P}{\alpha_0 P + N_2} \right) \quad (109)$$

$$= \frac{1}{2} \log \left(\frac{\alpha_1 P}{(9/5)N_2} \right). \quad (110)$$

Putting together,

$$\begin{aligned} T_1 &\geq \min \left\{ \frac{1}{2} \log \left(\frac{P}{\alpha_2 P} \cdot \frac{8}{21} \right), \frac{1}{2} \log \left(\frac{\alpha_1 P}{(9/5)N_2} \right) \right\} \\ &\quad + \frac{1}{2} \log \left(\frac{N_2}{N_1} \cdot \frac{4}{5} \right). \end{aligned}$$

Let us define α'_1 by the equality $\frac{1}{2} \log \left(\frac{P}{\alpha'_1 P} \cdot \frac{8}{21} \right) = \frac{1}{2} \log \left(\frac{\alpha_1 P}{(9/5)N_2} \right)$. If we choose $\alpha_2 \leq \alpha'_1$, then $\frac{1}{2} \log \left(\frac{P}{\alpha_2 P} \cdot \frac{8}{21} \right) \geq \frac{1}{2} \log \left(\frac{\alpha_1 P}{(9/5)N_2} \right)$, and

$$T_1 \geq \frac{1}{2} \log \left(\frac{\alpha_1 P}{(9/5)N_2} \cdot \frac{N_2}{N_1} \cdot \frac{4}{5} \right) = \frac{1}{2} \log \left(\frac{\alpha_1 P}{N_1} \cdot \frac{4}{9} \right).$$

We can see that the following rate region is achievable.

$$R_1 \leq \frac{1}{2} \log \left(\frac{\alpha_1 P}{N_1} \cdot \frac{4}{9} \right) \quad (111)$$

$$R_2 \leq \frac{1}{2} \log \left(\frac{\alpha_2 P}{(9/5)N_2} \right) \quad (112)$$

$$R_3 \leq \frac{1}{2} \log \left(\frac{P}{(4/3)\alpha_2 P} \right). \quad (113)$$

For fixed $\alpha_2 \in [\alpha_1, \alpha'_1]$ and $R_1 = \frac{1}{2} \log \left(\frac{\alpha_1 P}{N_1} \cdot \frac{4}{9} \right)$, the two-dimensional rate region \mathcal{R}_α , given by

$$R_2 \leq \frac{1}{2} \log \left(\frac{\alpha_2 P}{(9/5)N_2} \right) \quad (114)$$

$$R_3 \leq \frac{1}{2} \log \left(\frac{P}{(4/3)\alpha_2 P} \right) \quad (115)$$

is achievable. The union $\bigcup_{\alpha_2 \in [\alpha_1, \alpha'_1]} \mathcal{R}_\alpha$ is a MAC-like region, given by

$$R_2 \leq \frac{1}{2} \log \left(\frac{\alpha'_1 P}{(9/5)N_2} \right) \quad (116)$$

$$\leq \frac{1}{2} \log \left(\frac{P}{\alpha_1 P} \cdot \frac{8}{21} \right) \quad (117)$$

$$R_3 \leq \frac{1}{2} \log \left(\frac{P}{\alpha_1 P} \cdot \frac{3}{4} \right) \quad (118)$$

$$R_2 + R_3 \leq \frac{1}{2} \log \left(\frac{\alpha_2 P}{(9/5)N_2} \cdot \frac{P}{(4/3)\alpha_2 P} \right) \quad (119)$$

$$\leq \frac{1}{2} \log \left(\frac{P}{N_2} \cdot \frac{15}{36} \right). \quad (120)$$

This region is described in Fig. 6 (a).

In comparison, the two-dimensional outer bound region at $R_1 = \frac{1}{2} \log \left(\frac{\alpha_1 P}{N_1} \cdot \frac{4}{9} \right) + 1$, given by

$$R_2 \leq \frac{1}{2} \log \left(\frac{P}{N_1} \cdot \frac{7}{3} \right) - \frac{1}{2} \log \left(\frac{\alpha_1 P}{N_1} \cdot \frac{4}{9} \right) - 1$$

$$= \frac{1}{2} \log \left(\frac{P}{\alpha_1 P} \right) + \frac{1}{2} \log \left(\frac{7}{3} \cdot \frac{9}{4} \cdot \frac{1}{4} \right)$$

$$R_3 \leq \frac{1}{2} \log \left(\frac{P}{N_1} \cdot \frac{7}{3} \right) - \frac{1}{2} \log \left(\frac{\alpha_1 P}{N_1} \cdot \frac{4}{9} \right) - 1$$

$$= \frac{1}{2} \log \left(\frac{P}{\alpha_1 P} \right) + \frac{1}{2} \log \left(\frac{7}{3} \cdot \frac{9}{4} \cdot \frac{1}{4} \right)$$

$$R_2 + R_3 \leq \frac{1}{2} \log \left(\frac{P}{N_2} \right) + \frac{1}{2} \log \left(\frac{7}{3} \right).$$

Since $\delta_2 < \frac{1}{2} \log \left(\frac{21}{8} \cdot \frac{7}{3} \cdot \frac{9}{4} \cdot \frac{1}{4} \right) < 0.90$, $\delta_3 < \frac{1}{2} \log \left(\frac{4}{3} \cdot \frac{7}{3} \cdot \frac{9}{4} \cdot \frac{1}{4} \right) < 0.41$ and $\delta_{23} < \frac{1}{2} \log \left(\frac{36}{15} \cdot \frac{7}{3} \right) < 1.25 < \sqrt{2}$, we can conclude that the gap is to within one bit per message.

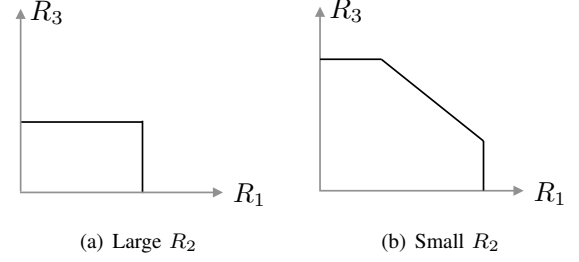


Fig. 7. The cross-section of the type 5 outer bound region at a relatively small or large R_2 .

VII. INNER BOUND: CHANNEL TYPE 5

Let us consider the relaxed outer bound region \mathcal{R}'_o given by

$$R_k \leq \frac{1}{2} \log \left(\frac{P}{N_k} \right) + \frac{1}{2} \log \left(\frac{4}{3} \right), \quad k = 1, 2, 3$$

$$R_1 + R_2 \leq \frac{1}{2} \log \left(\frac{P}{N_1} \right) + \frac{1}{2} \log \left(\frac{7}{3} \right)$$

$$R_2 + R_3 \leq \frac{1}{2} \log \left(\frac{P}{N_2} \right) + \frac{1}{2} \log \left(\frac{7}{3} \right)$$

$$R_1 + R_3 \leq \frac{1}{2} \log \left(\frac{P}{N_1} \right) + \frac{1}{2} \log \left(\frac{7}{3} \right).$$

The cross-sectional region at a given R_2 is described by

$$R_1 \leq \min \left\{ \frac{1}{2} \log \left(\frac{P}{N_1} \cdot \frac{7}{3} \right) - R_2, \frac{1}{2} \log \left(\frac{P}{N_1} \cdot \frac{4}{3} \right) \right\}$$

$$R_3 \leq \min \left\{ \frac{1}{2} \log \left(\frac{P}{N_2} \cdot \frac{7}{3} \right) - R_2, \frac{1}{2} \log \left(\frac{P}{N_3} \cdot \frac{4}{3} \right) \right\}$$

$$R_1 + R_3 \leq \frac{1}{2} \log \left(\frac{P}{N_1} \cdot \frac{7}{3} \right).$$

Depending on the bottleneck of $\min\{\cdot, \cdot\}$ expressions, there are three cases:

- $R_2 \leq \frac{1}{2} \log \left(\frac{7}{4} \right)$
- $\frac{1}{2} \log \left(\frac{7}{4} \right) \leq R_2 \leq \frac{1}{2} \log \left(\frac{N_3}{N_2} \cdot \frac{7}{4} \right)$
- $R_2 \geq \frac{1}{2} \log \left(\frac{N_3}{N_2} \cdot \frac{7}{4} \right)$.

In this section, we focus on the third case. The other cases can be proved similarly. If the sum of the righthand sides of R_1 and R_3 bounds is smaller than the righthand side of $R_1 + R_3$ bound, i.e.,

$$\frac{1}{2} \log \left(\frac{P}{N_1} \cdot \frac{7}{3} \right) + \frac{1}{2} \log \left(\frac{P}{N_2} \cdot \frac{7}{3} \right) - 2R_2 \leq \frac{1}{2} \log \left(\frac{P}{N_1} \cdot \frac{7}{3} \right),$$

then the $R_1 + R_3$ bound is not active at the R_2 . By rearranging, the threshold condition is given by

$$R_2 > R_{2,th} = \frac{1}{4} \log \left(\frac{P}{N_2} \cdot \frac{7}{3} \right). \quad (121)$$

Note that $R_{2,th}$ is roughly half of C_2 . For this relatively large R_2 , the cross-sectional region is a rectangle as described in Fig. 7 (a). In contrast, for a relatively small R_1 , when the threshold condition does not hold, the cross-sectional region is a MAC-like region as described in Fig. 7 (b). In the following subsections, we present achievable schemes for each case.

A. Achievable Scheme for Relatively Large R_2

Theorem 13: Given $\alpha = (\alpha_1, \alpha_2, \alpha'_2) \in [0, 1]^3$, the region \mathcal{R}_α is defined by

$$\begin{aligned} R_1 &\leq \frac{1}{2} \log \left(1 + \frac{\alpha_1 P}{N_1} \right) \\ R_2 &\leq \min \left\{ \frac{1}{2} \log^+ \left(c_{21} + \frac{(1 - \alpha_2 - \alpha'_2)P}{(\alpha_1 + \alpha_2 + \alpha'_2)P + N_2} \right), \right. \\ &\quad \left. \frac{1}{2} \log \left(1 + \frac{\alpha'_2 P}{N_2} \right) \right\} + \frac{1}{2} \log \left(1 + \frac{\alpha_2 P}{\alpha'_2 P + N_2} \right) \\ R_3 &\leq \frac{1}{2} \log^+ \left(c_3 + \frac{P}{\max\{\alpha_1, \alpha_2 + \alpha'_2\}P + N_3} \right) \end{aligned}$$

where $c_{21} = \frac{1 - \alpha_2 - \alpha'_2}{2 - \alpha_2 - \alpha'_2}$ and $c_3 = \frac{1}{2 - \alpha_2 - \alpha'_2}$, and $\mathcal{R} = \text{CONV}(\bigcup_\alpha \mathcal{R}_\alpha)$ is achievable.

We present an achievable scheme for the case of $R_2 > R_{2,th}$. Message $M_2 \in \{1, 2, \dots, 2^{nR_2}\}$ for receiver 2 is split into two parts: $M_{21} \in \{1, 2, \dots, 2^{nR_{21}}\}$ and $M_{22} \in \{1, 2, \dots, 2^{nR_{22}}\}$, so $R_2 = R_{21} + R_{22}$. We generate the signals in the following way: \mathbf{x}_{21} and \mathbf{x}'_{21} are differently coded signals of M_{21} , and \mathbf{x}_{22} is a coded signal of M_{22} . The transmit signal is the sum

$$\mathbf{x}_2 = \mathbf{x}_{21} + \mathbf{x}_{22} + \mathbf{x}'_{21}.$$

The power allocation satisfies $\mathbb{E}[\|\mathbf{x}_1\|^2] = \alpha_1 nP$, at receiver 1, $\mathbb{E}[\|\mathbf{x}_{21}\|^2] = \alpha'_2 nP$, $\mathbb{E}[\|\mathbf{x}_{22}\|^2] = \alpha_2 nP$, and $\mathbb{E}[\|\mathbf{x}'_{21}\|^2] = (1 - \alpha_2 - \alpha'_2)P$ at receiver 2, and $\mathbb{E}[\|\mathbf{x}_3\|^2] = nP$ at receiver 3.

The signals \mathbf{x}'_{21} and \mathbf{x}_3 are lattice codewords using the same coding lattice but different shaping lattices. As a result, the sum $\mathbf{x}'_{21} + \mathbf{x}_3$ is a lattice codeword.

The received signals are

$$\begin{aligned} \mathbf{y}_1 &= \mathbf{x}'_{21} + \mathbf{x}_{22} + \mathbf{x}_{21} + \mathbf{x}_1 + \mathbf{z}_1 \\ \mathbf{y}_2 &= [\mathbf{x}'_{21} + \mathbf{x}_3] + \mathbf{x}_{22} + \mathbf{x}_{21} + \mathbf{z}_2 \\ \mathbf{y}_3 &= \mathbf{x}_3 + \mathbf{x}_1 + \mathbf{z}_3. \end{aligned}$$

The signal scale diagram at each receiver is shown in Fig. 8 (a). Decoding is performed in the following way.

- At receiver 1, \mathbf{x}'_{21} is first decoded while treating other signals as noise. Having successfully recovered M_{21} , receiver 1 can generate \mathbf{x}_{21} and \mathbf{x}'_{21} , and cancel them from \mathbf{y}_1 . Next, \mathbf{x}_{22} is decoded from $\mathbf{x}_{22} + \mathbf{x}_1 + \mathbf{z}_1$. Finally, \mathbf{x}_1 is decoded from $\mathbf{x}_1 + \mathbf{z}_1$. For reliable decoding, the code rates should satisfy

$$\begin{aligned} R_{21} &\leq T'_{21} = \frac{1}{2} \log \left(1 + \frac{(1 - \alpha_2 - \alpha'_2)P}{(\alpha_1 + \alpha_2 + \alpha'_2)P + N_1} \right) \\ R_{22} &\leq T'_{22} = \frac{1}{2} \log \left(1 + \frac{\alpha_2 P}{\alpha_1 P + N_1} \right) \\ R_1 &\leq T_1 = \frac{1}{2} \log \left(1 + \frac{\alpha_1 P}{N_1} \right). \end{aligned}$$

- At receiver 2, $[\mathbf{x}'_{21} + \mathbf{x}_3]$ first decoded while treating other signals as noise and removed from \mathbf{y}_2 . Next, \mathbf{x}_{22} and \mathbf{x}_{21}

are decoded successively. For reliable decoding, the code rates should satisfy

$$\begin{aligned} R_{21} &\leq T''_{21} = \frac{1}{2} \log \left(c_{21} + \frac{(1 - \alpha_2 - \alpha'_2)P}{(\alpha_2 + \alpha'_2)P + N_2} \right) \\ R_3 &\leq T'_3 = \frac{1}{2} \log \left(c_3 + \frac{P}{(\alpha_2 + \alpha'_2)P + N_2} \right) \\ R_{22} &\leq T''_{22} = \frac{1}{2} \log \left(1 + \frac{\alpha_2 P}{\alpha'_2 P + N_2} \right) \\ R_{21} &\leq T'''_{21} = \frac{1}{2} \log \left(1 + \frac{\alpha'_2 P}{N_2} \right) \end{aligned}$$

where $c_{21} = \frac{(1 - \alpha_2 - \alpha'_2)P}{(1 - \alpha_2 - \alpha'_2)P + P} = \frac{1 - \alpha_2 - \alpha'_2}{2 - \alpha_2 - \alpha'_2}$ and $c_3 = \frac{P}{(1 - \alpha_2 - \alpha'_2)P + P} = \frac{1}{2 - \alpha_2 - \alpha'_2}$. Note that $0 \leq c_{21} \leq \frac{1}{2}$, $c_{21} + c_3 = 1$, and $\frac{1}{2} \leq c_3 \leq 1$.

- At receiver 3, \mathbf{x}_3 is decoded while treating $\mathbf{x}_1 + \mathbf{z}_3$ as noise. Reliable decoding is possible if

$$R_3 \leq T'_3 = \frac{1}{2} \log \left(1 + \frac{P}{\alpha_1 P + N_3} \right). \quad (122)$$

Putting together, we can see that given $\alpha_1, \alpha_2, \alpha'_2 \in [0, 1]$, the following rate region is achievable.

$$\begin{aligned} R_1 &\leq T_1 \\ R_2 &\leq T_2 = \min\{T'_{21}, T''_{21}, T'''_{21}\} + \min\{T'_{22}, T''_{22}\} \\ R_3 &\leq T_3 = \min\{T'_3, T''_3\} \end{aligned}$$

where

$$\begin{aligned} T_1 &= \frac{1}{2} \log \left(1 + \frac{\alpha_1 P}{N_1} \right) \\ T_2 &= \min\{T'_{21}, T''_{21}, T'''_{21}\} + T''_{22} \\ &= \min\{\min\{T'_{21}, T''_{21}\}, T'''_{21}\} + T''_{22} \\ &\geq \min \left\{ \frac{1}{2} \log \left(c_{21} + \frac{(1 - \alpha_2 - \alpha'_2)P}{(\alpha_1 + \alpha_2 + \alpha'_2)P + N_2} \right), \right. \\ &\quad \left. \frac{1}{2} \log \left(1 + \frac{\alpha'_2 P}{N_2} \right) \right\} + \frac{1}{2} \log \left(1 + \frac{\alpha_2 P}{\alpha'_2 P + N_2} \right) \\ T_3 &\geq \frac{1}{2} \log \left(c_3 + \frac{P}{\max\{\alpha_1, \alpha_2 + \alpha'_2\}P + N_3} \right). \end{aligned}$$

B. The Gap for Relatively Large R_2

We choose α_1 and α_2 such that $\alpha_1 P \geq N_2$, that $\alpha_2 P \geq N_3$, that $\alpha_1 = \alpha'_2 \leq \alpha_2$, and that $\alpha_1 + \alpha_2 \leq \frac{1}{2}$. It follows that $c_{21} \geq \frac{1}{3}$. We get the lower bounds for each term of T_2 expression above.

$$\min\{T'_{21}, T''_{21}\} \quad (123)$$

$$\geq \frac{1}{2} \log \left(c_{21} + \frac{(1 - \alpha_1 - \alpha_2)P}{(2\alpha_1 + \alpha_2)P + N_2} \right) \quad (124)$$

$$\geq \frac{1}{2} \log \left(\frac{1}{3} + \frac{(1 - \alpha_1 - \alpha_2)P}{(3\alpha_1 + \alpha_2)P} \right) \quad (125)$$

$$\geq \frac{1}{2} \log \left(\frac{P}{(3\alpha_1 + \alpha_2)P} \right). \quad (126)$$

The first entry of $\min\{\cdot, \cdot\}$ in

$$T_2 = \min\{\min\{T'_{21}, T''_{21}\} + T''_{22}, T'''_{21} + T''_{22}\}$$

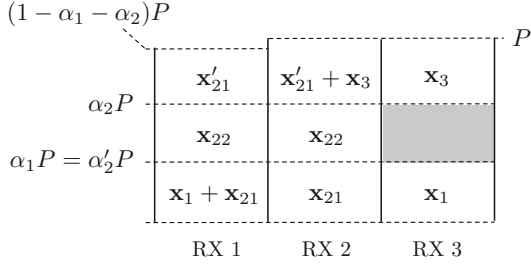
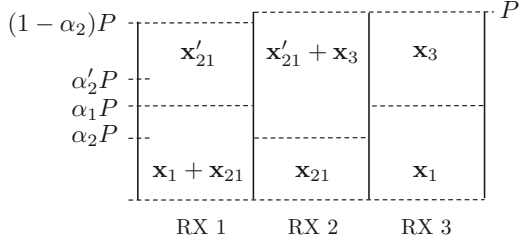
(a) Channel type 5: relatively large R_2 (b) Channel type 5: relatively small R_2

Fig. 8. Signal scale diagram.

is lower bounded as follows.

$$\begin{aligned}
 & \min\{T'_{21}, T''_{21}\} + T''_{22} \\
 & \geq \frac{1}{2} \log \left(\frac{P}{(3\alpha_1 + \alpha_2)P} \right) + \frac{1}{2} \log \left(\frac{(\alpha_1 + \alpha_2)P + N_2}{\alpha_1 P + N_2} \right) \\
 & = \frac{1}{2} \log \left(\frac{P}{\alpha_1 P + N_2} \cdot \frac{(\alpha_1 + \alpha_2)P + N_2}{(3\alpha_1 + \alpha_2)P} \right) \\
 & \geq \frac{1}{2} \log \left(\frac{P}{3(\alpha_1 P + N_2)} \right) \\
 & \geq \frac{1}{2} \log \left(\frac{P}{6\alpha_1 P} \right).
 \end{aligned}$$

The second entry of $T_2 = \min\{\cdot, \cdot\}$ is lower bounded as follows.

$$\begin{aligned}
 & T'''_{21} + T''_{22} \\
 & = \frac{1}{2} \log \left(1 + \frac{\alpha_1 P}{N_2} \right) + \frac{1}{2} \log \left(1 + \frac{\alpha_2 P}{\alpha_1 P + N_2} \right) \\
 & = \frac{1}{2} \log \left(1 + \frac{(\alpha_1 + \alpha_2)P}{N_2} \right) \\
 & \geq \frac{1}{2} \log \left(\frac{\alpha_2 P}{N_2} \right).
 \end{aligned}$$

Putting together, we get the lower bound

$$T_2 \geq \min \left\{ \frac{1}{2} \log \left(\frac{P}{6\alpha_1 P} \right), \frac{1}{2} \log \left(\frac{\alpha_2 P}{N_2} \right) \right\}.$$

Given α_2 , we choose α_1 that satisfies $\frac{1}{2} \log \left(\frac{P}{6\alpha_1 P} \right) = \frac{1}{2} \log \left(\frac{\alpha_2 P}{N_2} \right)$. As a result, we can write $T_2 \geq \frac{1}{2} \log \left(\frac{\alpha_2 P}{N_2} \right)$. We also have

$$T_3 \geq \frac{1}{2} \log \left(\frac{P}{(\alpha_1 + \alpha_2)P + N_3} \right) \geq \frac{1}{2} \log \left(\frac{P}{3\alpha_2 P} \right).$$

Putting together, we can see that the following rate region is achievable.

$$R_1 \leq \frac{1}{2} \log \left(\frac{\alpha_1 P}{N_1} \right) \quad (127)$$

$$R_2 \leq \frac{1}{2} \log \left(\frac{\alpha_2 P}{N_2} \right) \quad (128)$$

$$R_3 \leq \frac{1}{2} \log \left(\frac{P}{3\alpha_2 P} \right). \quad (129)$$

For fixed α_2 and $R_2 = \frac{1}{2} \log \left(\frac{\alpha_2 P}{N_2} \right)$, the two-dimensional rate region, given by

$$R_1 \leq \frac{1}{2} \log \left(\frac{\alpha_1 P}{N_1} \right) \quad (130)$$

$$= \frac{1}{2} \log \left(\frac{P}{6\alpha_2 P} \cdot \frac{N_2}{N_1} \right) \quad (131)$$

$$R_3 \leq \frac{1}{2} \log \left(\frac{P}{3\alpha_2 P} \right) \quad (132)$$

is achievable.

In comparison, the two-dimensional outer bound region at $R_2 = \frac{1}{2} \log \left(\frac{\alpha_2 P}{N_2} \right) + 1$ is given by

$$\begin{aligned}
 R_1 & \leq \frac{1}{2} \log \left(\frac{P}{N_1} \cdot \frac{7}{3} \right) - \frac{1}{2} \log \left(\frac{\alpha_2 P}{N_2} \right) - 1 \\
 & = \frac{1}{2} \log \left(\frac{P}{\alpha_2 P} \cdot \frac{N_2}{N_1} \right) + \frac{1}{2} \log \left(\frac{7}{3} \cdot \frac{1}{4} \right) \\
 R_3 & \leq \frac{1}{2} \log \left(\frac{P}{N_2} \cdot \frac{7}{3} \right) - \frac{1}{2} \log \left(\frac{\alpha_2 P}{N_2} \right) - 1 \\
 & = \frac{1}{2} \log \left(\frac{P}{\alpha_2 P} \right) + \frac{1}{2} \log \left(\frac{7}{3} \cdot \frac{1}{4} \right).
 \end{aligned}$$

As discussed above, the sum-rate bound on $R_1 + R_3$ is loose for R_2 larger than the threshold, so the rate region is a rectangle.

By comparing the inner and outer bound rate regions, we can see that $\delta_1 < \frac{1}{2} \log \left(6 \cdot \frac{7}{3} \cdot \frac{1}{4} \right) < 0.91$ and $\delta_3 < \frac{1}{2} \log \left(3 \cdot \frac{7}{3} \cdot \frac{1}{4} \right) < 0.41$. Therefore, we can conclude that the gap is to within one bit per message.

C. Achievable Scheme for Relatively Small R_2

Theorem 14: Given $\alpha = (\alpha_1, \alpha_2) \in [0, 1]^2$, the region \mathcal{R}_α is defined by

$$\begin{aligned}
 R_1 & \leq \frac{1}{2} \log \left(1 + \frac{\alpha_1 P}{N_1} \right) \\
 R_2 & \leq \min \left\{ \frac{1}{2} \log^+ \left(c_{21} + \frac{(1 - \alpha_2)P}{(\alpha_1 + \alpha_2)P + N_2} \right), \right. \\
 & \quad \left. \frac{1}{2} \log \left(1 + \frac{\alpha_2 P}{N_2} \right) \right\} \\
 R_3 & \leq \frac{1}{2} \log^+ \left(c_3 + \frac{P}{\max\{\alpha_1, \alpha_2\}P + N_3} \right)
 \end{aligned}$$

where $c_{21} = \frac{1 - \alpha_2}{2 - \alpha_2}$ and $c_3 = \frac{1}{2 - \alpha_2}$, and $\mathcal{R} = \text{CONV}(\bigcup_\alpha \mathcal{R}_\alpha)$ is achievable.

For the case of $R_2 < R_{2,th}$, we present the following scheme. At transmitter 2, rate splitting is not necessary. The transmit signal is the sum

$$\mathbf{x}_2 = \mathbf{x}_{21} + \mathbf{x}'_{21}$$

where \mathbf{x}_{21} and \mathbf{x}'_{21} are differently coded versions of the same message $M_2 \in \{1, 2, \dots, 2^{nR_2}\}$.

The power allocation: $\mathbb{E}[\|\mathbf{x}_1\|^2] = \alpha_1 nP$ at receiver 1, $\mathbb{E}[\|\mathbf{x}_{21}\|^2] = \alpha_2 nP$, and $\mathbb{E}[\|\mathbf{x}'_{21}\|^2] = (1 - \alpha_2)nP$ at receiver 2, and $\mathbb{E}[\|\mathbf{x}_3\|^2] = nP$ at receiver 3.

The signals \mathbf{x}'_{21} and \mathbf{x}_3 are lattice codewords using the same coding lattice but different shaping lattices. As a result, the sum $\mathbf{x}'_{21} + \mathbf{x}_3$ is a lattice codeword.

The received signals are

$$\begin{aligned} \mathbf{y}_1 &= \mathbf{x}'_{21} + \mathbf{x}_{21} + \mathbf{x}_1 + \mathbf{z}_1 \\ \mathbf{y}_2 &= [\mathbf{x}'_{21} + \mathbf{x}_3] + \mathbf{x}_{21} + \mathbf{z}_2 \\ \mathbf{y}_3 &= \mathbf{x}_3 + \mathbf{x}_1 + \mathbf{z}_3. \end{aligned}$$

The signal scale diagram at each receiver is shown in Fig. 8 (b). Decoding is performed in the following way.

- At receiver 1, \mathbf{x}'_{21} is first decoded while treating other signals as noise. Having successfully recovered M_{21} , receiver 1 can generate \mathbf{x}_{21} and \mathbf{x}'_{21} , and cancel them from \mathbf{y}_1 . Next, \mathbf{x}_1 is decoded from $\mathbf{x}_1 + \mathbf{z}_1$. For reliable decoding, the code rates should satisfy

$$\begin{aligned} R_{21} &\leq T'_{21} = \frac{1}{2} \log \left(1 + \frac{(1 - \alpha_2)P}{(\alpha_1 + \alpha_2)P + N_1} \right) \\ R_1 &\leq T_1 = \frac{1}{2} \log \left(1 + \frac{\alpha_1 P}{N_1} \right). \end{aligned}$$

- At receiver 2, $[\mathbf{x}'_{21} + \mathbf{x}_3]$ first decoded while treating other signals as noise and removed from \mathbf{y}_2 . Next, \mathbf{x}_{21} is decoded from $\mathbf{x}_{21} + \mathbf{z}_2$. For reliable decoding, the code rates should satisfy

$$\begin{aligned} R_{21} &\leq T''_{21} = \frac{1}{2} \log \left(c_{21} + \frac{(1 - \alpha_2)P}{\alpha_2 P + N_2} \right) \\ R_3 &\leq T'_3 = \frac{1}{2} \log \left(c_3 + \frac{P}{\alpha_2 P + N_2} \right) \\ R_{21} &\leq T'''_{21} = \frac{1}{2} \log \left(1 + \frac{\alpha_2 P}{N_2} \right) \end{aligned}$$

where $c_{21} = \frac{(1 - \alpha_2)P}{(1 - \alpha_2)P + P} = \frac{1 - \alpha_2}{2 - \alpha_2}$ and $c_3 = \frac{P}{(1 - \alpha_2)P + P} = \frac{1}{2 - \alpha_2}$. Note that $0 \leq c_{21} \leq \frac{1}{2}$, $c_{21} + c_3 = 1$, and $\frac{1}{2} \leq c_3 \leq 1$.

- At receiver 3, \mathbf{x}_3 is decoded while treating $\mathbf{x}_1 + \mathbf{z}_3$ as noise. Reliable decoding is possible if

$$R_3 \leq T''_3 = \frac{1}{2} \log \left(1 + \frac{P}{\alpha_1 P + N_3} \right). \quad (133)$$

Putting together, we get

$$R_1 \leq T_1 \quad (134)$$

$$R_2 \leq T_2 = \min\{T'_{21}, T''_{21}, T'''_{21}\} \quad (135)$$

$$R_3 \leq T_3 = \min\{T'_3, T''_3\} \quad (136)$$

where

$$\begin{aligned} T_1 &= \frac{1}{2} \log \left(1 + \frac{\alpha_1 P}{N_1} \right) \\ T_2 &= \min\{T'_{21}, T''_{21}, T'''_{21}\} \\ &= \min\{\min\{T'_{21}, T''_{21}\}, T'''_{21}\} \\ &\geq \min \left\{ \frac{1}{2} \log \left(c_{21} + \frac{(1 - \alpha_2)P}{(\alpha_1 + \alpha_2)P + N_2} \right), \right. \\ &\quad \left. \frac{1}{2} \log \left(1 + \frac{\alpha_2 P}{N_2} \right) \right\} \\ T_3 &\geq \frac{1}{2} \log \left(c_3 + \frac{P}{\max\{\alpha_1, \alpha_2\}P + N_3} \right). \end{aligned}$$

D. The Gap for Relatively Small R_2

We choose α_1 and α_2 such that $\alpha_1 P \geq N_2$, that $\alpha_2 P \geq N_3$, that $\alpha_1 + \alpha_2 \leq \frac{1}{2}$, and that $\alpha_1 \geq \alpha_2$. It follows that $c_{21} \geq \frac{1}{3}$. We get the lower bound

$$\min\{T'_{21}, T''_{21}\} \quad (137)$$

$$= \frac{1}{2} \log \left(c_{21} + \frac{(1 - \alpha_2)P}{(\alpha_1 + \alpha_2)P + N_2} \right) \quad (138)$$

$$\geq \frac{1}{2} \log \left(\frac{1}{3} + \frac{(1 - \alpha_1)P}{3\alpha_1 P} \right) \quad (139)$$

$$= \frac{1}{2} \log \left(\frac{P}{3\alpha_1 P} \right) \quad (140)$$

and

$$T_2 \geq \min \left\{ \frac{1}{2} \log \left(\frac{P}{3\alpha_1 P} \right), \frac{1}{2} \log \left(\frac{\alpha_2 P}{N_2} \right) \right\}.$$

Let us define α'_2 by the equality $\frac{1}{2} \log \left(\frac{P}{3\alpha'_2 P} \right) = \frac{1}{2} \log \left(\frac{\alpha_2 P}{N_2} \right)$. If we choose $\alpha_1 \leq \alpha'_2$, then $T_2 \geq \frac{1}{2} \log \left(\frac{\alpha_2 P}{N_2} \right)$. We can see that the following rate region is achievable.

$$R_1 \leq \frac{1}{2} \log \left(\frac{\alpha_1 P}{N_1} \right) \quad (141)$$

$$R_2 \leq \frac{1}{2} \log \left(\frac{\alpha_2 P}{N_2} \right) \quad (142)$$

$$R_3 \leq \frac{1}{2} \log \left(\frac{P}{2\alpha_1 P} \right). \quad (143)$$

For fixed $\alpha_1 \in [\alpha_2, \alpha'_2]$ and $R_2 = \frac{1}{2} \log \left(\frac{\alpha_2 P}{N_2} \right)$, the two-dimensional rate region \mathcal{R}_α , given by

$$R_1 \leq \frac{1}{2} \log \left(\frac{\alpha_1 P}{N_1} \right) \quad (144)$$

$$R_3 \leq \frac{1}{2} \log \left(\frac{P}{2\alpha_1 P} \right) \quad (145)$$

is achievable. The union $\bigcup_{\alpha_1 \in [\alpha_2, \alpha'_2]} \mathcal{R}_\alpha$ is a MAC-like region, given by

$$R_1 \leq \frac{1}{2} \log \left(\frac{\alpha'_2 P}{N_1} \right) \quad (146)$$

$$= \frac{1}{2} \log \left(\frac{P}{3\alpha_2 P} \cdot \frac{N_2}{N_1} \right) \quad (147)$$

$$R_3 \leq \frac{1}{2} \log \left(\frac{P}{2\alpha_2 P} \right) \quad (148)$$

$$R_1 + R_3 \leq \frac{1}{2} \log \left(\frac{P}{2N_1} \right). \quad (149)$$

In comparison, the two-dimensional outer bound region at $R_2 = \frac{1}{2} \log \left(\frac{\alpha_2 P}{N_2} \right) + 1$ is given by

$$\begin{aligned} R_1 &\leq \frac{1}{2} \log \left(\frac{P}{N_1} \cdot \frac{7}{3} \right) - \frac{1}{2} \log \left(\frac{\alpha_2 P}{N_2} \right) - 1 \\ &= \frac{1}{2} \log \left(\frac{P}{\alpha_2 P} \cdot \frac{N_2}{N_1} \right) + \frac{1}{2} \log \left(\frac{7}{3} \cdot \frac{1}{4} \right) \\ R_3 &\leq \frac{1}{2} \log \left(\frac{P}{N_2} \cdot \frac{7}{3} \right) - \frac{1}{2} \log \left(\frac{\alpha_2 P}{N_2} \right) - 1 \\ &= \frac{1}{2} \log \left(\frac{P}{\alpha_2 P} \right) + \frac{1}{2} \log \left(\frac{7}{3} \cdot \frac{1}{4} \right) \\ R_1 + R_3 &\leq \frac{1}{2} \log \left(\frac{P}{N_1} \cdot \frac{8}{3} \right). \end{aligned}$$

Since $\delta_1 < \frac{1}{2} \log \left(3 \cdot \frac{7}{3} \cdot \frac{1}{4} \right) < 0.41$, $\delta_3 < \frac{1}{2} \log \left(2 \cdot \frac{7}{3} \cdot \frac{1}{4} \right) < 0.12$ and $\delta_{13} < \frac{1}{2} \log \left(2 \cdot \frac{7}{3} \right) < 1.12 < \sqrt{2}$, we can conclude that the gap is to within one bit per message.

VIII. CONCLUSION

We presented approximate capacity region of five important cases of partially connected interference channels. The outer bounds based on Z -channel type argument are derived. Achievable schemes are developed and shown to approximately achieve the capacity to within a constant bit.

For future work, the channels with fully general coefficients may be considered. In this paper, we presented different schemes for each channel type although they share some principle. A universal scheme is to be developed for unified capacity characterization of all possible topologies. The connection between interference channel and index coding problems is much to explore. In particular, the results on the capacity region for index coding in [23] seem to have an interesting connection to our work.

APPENDIX A

RANDOM CODING ACHIEVABILITY: CHANNEL TYPE 4

At transmitter 1, message M_1 is split into three parts (M_{12}, M_{11}, M_{10}) , and the transmit signal is $\mathbf{x}_1 = \mathbf{x}_{12} + \mathbf{x}_{11} + \mathbf{x}_{10}$. The signals satisfy $\mathbb{E}[\|\mathbf{x}_{12}\|^2] = n(P - N_2 - N_3)$, $\mathbb{E}[\|\mathbf{x}_{11}\|^2] = nN_3$, and $\mathbb{E}[\|\mathbf{x}_{10}\|^2] = nN_2$.

At transmitter 2, message M_2 is split into three parts (M_{21}, M_{20}) , and the transmit signal is $\mathbf{x}_2 = \mathbf{x}_{21} + \mathbf{x}_{20}$. The signals satisfy $\mathbb{E}[\|\mathbf{x}_{21}\|^2] = n(P - N_3)$ and $\mathbb{E}[\|\mathbf{x}_{20}\|^2] = nN_3$. Rate-splitting is not performed at transmitter 3, and $\mathbb{E}[\|\mathbf{x}_3\|^2] = nP$.

The top layer codewords $(\mathbf{x}_{12}, \mathbf{x}_{21}, \mathbf{x}_3)$ are from a joint random codebook for (M_{12}, M_{21}, M_3) . The mid-layer codewords $(\mathbf{x}_{11}, \mathbf{x}_{20})$ are from a joint random codebook for (M_{11}, M_{20}) . The bottom layer codeword \mathbf{x}_{10} is from a single-user random codebook for M_{10} .

The received signals are

$$\begin{aligned} \mathbf{y}_1 &= (\mathbf{x}_{12} + \mathbf{x}_3) + \mathbf{x}_{11} + \mathbf{x}_{10} + \mathbf{z}_1 \\ \mathbf{y}_2 &= (\mathbf{x}_{12} + \mathbf{x}_{21}) + (\mathbf{x}_{11} + \mathbf{x}_{20}) + \mathbf{x}_{10} + \mathbf{z}_2 \\ \mathbf{y}_3 &= (\mathbf{x}_{21} + \mathbf{x}_3) + \mathbf{x}_{20} + \mathbf{z}_3 \end{aligned}$$

Decoding is performed from the top layer to the bottom layer. At receiver 1, simultaneous decoding of $(\mathbf{x}_{12}, \mathbf{x}_3)$ is performed

while treating other signals as noise. And then, \mathbf{x}_{11} and \mathbf{x}_{10} are decoded successively. At receiver 2, simultaneous decoding of $(\mathbf{x}_{12}, \mathbf{x}_{21})$ is performed while treating other signals as noise. And then, simultaneous decoding of $(\mathbf{x}_{11}, \mathbf{x}_{20})$ is performed. At receiver 3, simultaneous decoding of $(\mathbf{x}_{21}, \mathbf{x}_3)$ is performed while treating other signals as noise. For reliable decoding, code rates should satisfy

$$\begin{aligned} R_{12} &\leq I_1 = \frac{1}{2} \log \left(1 + \frac{P - N_2 - N_3}{N_1 + N_2 + N_3} \right) \\ R_3 &\leq I_2 = \frac{1}{2} \log \left(1 + \frac{P}{N_1 + N_2 + N_3} \right) \\ R_{12} + R_3 &\leq I_3 = \frac{1}{2} \log \left(1 + \frac{2P - N_2 - N_3}{N_1 + N_2 + N_3} \right) \\ R_{11} &\leq I_4 = \frac{1}{2} \log \left(1 + \frac{N_3}{N_1 + N_2} \right) \\ R_{10} &\leq I_5 = \frac{1}{2} \log \left(1 + \frac{N_2}{N_1} \right) \end{aligned}$$

at receiver 1,

$$\begin{aligned} R_{12} &\leq I_6 = \frac{1}{2} \log \left(1 + \frac{P - N_2 - N_3}{2N_2 + 2N_3} \right) \\ R_{21} &\leq I_7 = \frac{1}{2} \log \left(1 + \frac{P - N_3}{2N_2 + 2N_3} \right) \\ R_{12} + R_{21} &\leq I_8 = \frac{1}{2} \log \left(1 + \frac{2P - N_2 - 2N_3}{2N_2 + 2N_3} \right) \\ R_{11} &\leq I_9 = \frac{1}{2} \log \left(1 + \frac{N_3}{2N_2} \right) \\ R_{20} &\leq I_{10} = \frac{1}{2} \log \left(1 + \frac{N_3}{2N_2} \right) \\ R_{11} + R_{20} &\leq I_{11} = \frac{1}{2} \log \left(1 + \frac{2N_3}{2N_2} \right) \end{aligned}$$

at receiver 2,

$$\begin{aligned} R_{21} &\leq I_{12} = \frac{1}{2} \log \left(1 + \frac{P - N_3}{2N_3} \right) \\ R_3 &\leq I_{13} = \frac{1}{2} \log \left(1 + \frac{P}{2N_3} \right) \\ R_{21} + R_3 &\leq I_{14} = \frac{1}{2} \log \left(1 + \frac{2P - N_3}{2N_3} \right) \end{aligned}$$

at receiver 3. Putting together,

$$\begin{aligned} R_{12} &\leq T_1 = \min\{I_1, I_6\} = I_6 \\ R_{21} &\leq T_2 = \min\{I_7, I_{12}\} = I_7 \\ R_3 &\leq T_3 = \min\{I_2, I_{13}\} \\ R_{12} + R_{21} &\leq T_4 = I_8 \\ R_{12} + R_3 &\leq T_5 = I_3 \\ R_{21} + R_3 &\leq T_6 = I_{14} \end{aligned}$$

at the top layer,

$$\begin{aligned} R_{11} &\leq T_7 = \min\{I_4, I_9\} = I_9 \\ R_{20} &\leq T_8 = I_{10} \\ R_{11} + R_{20} &\leq T_9 = I_{11} \end{aligned}$$

at the mid-layer,

$$R_{10} \leq T_{10} = I_5$$

at the bottom layer. Note that the rate variables are not coupled between layers. We get the achievable rate region

$$\begin{aligned} R_1 &= R_{12} + R_{11} + R_{10} \leq T_1 + T_7 + T_{10} \\ R_2 &= R_{21} + R_{20} \leq T_2 + T_8 \\ R_3 &\leq T_3 \\ R_1 + R_2 &\leq T_4 + T_9 + T_{10} \\ R_1 + R_3 &\leq T_5 + T_7 + T_{10} \\ R_2 + R_3 &\leq T_6 + T_8. \end{aligned}$$

This region includes the following region.

$$\begin{aligned} R_1 &\leq \frac{1}{2} \log \left(2 + \frac{P}{N_1} \right) - 1 \\ R_2 &\leq \frac{1}{2} \log \left(3 + \frac{P}{N_2} \right) - 1 \\ R_3 &\leq \frac{1}{2} \log \left(3 + \frac{P}{N_3} \right) - \frac{1}{2} \log(3) \\ R_1 + R_2 &\leq \frac{1}{2} \log \left(1 + \frac{2P}{N_1} \right) - \frac{1}{2} \\ R_1 + R_3 &\leq \frac{1}{2} \log \left(1 + \frac{2P}{N_1} \right) - 1 \\ R_2 + R_3 &\leq \frac{1}{2} \log \left(1 + \frac{2P}{N_2} \right) - 1. \end{aligned}$$

Therefore, we can conclude the capacity region to within one bit.

APPENDIX B

RANDOM CODING ACHIEVABILITY: CHANNEL TYPE 5

Transmit signal construction is the same as the one for channel type 4. The received signals are

$$\begin{aligned} \mathbf{y}_1 &= (\mathbf{x}_{12} + \mathbf{x}_{21}) + (\mathbf{x}_{11} + \mathbf{x}_{20}) + \mathbf{x}_{10} + \mathbf{z}_1 \\ \mathbf{y}_2 &= (\mathbf{x}_{21} + \mathbf{x}_3) + \mathbf{x}_{20} + \mathbf{z}_2 \\ \mathbf{y}_3 &= (\mathbf{x}_{12} + \mathbf{x}_3) + \mathbf{x}_{11} + \mathbf{x}_{10} + \mathbf{z}_3 \end{aligned}$$

Decoding is performed from the top layer to the bottom layer. At receiver 1, simultaneous decoding of $(\mathbf{x}_{12}, \mathbf{x}_{21})$ is performed while treating other signals as noise. And then, simultaneous decoding of \mathbf{x}_{11} and \mathbf{x}_{20} is performed. Lastly, \mathbf{x}_{10} is decoded. At receiver 2, simultaneous decoding of $(\mathbf{x}_{21}, \mathbf{x}_3)$ is performed while treating other signals as noise. And then, \mathbf{x}_{20} is decoded. At receiver 3, simultaneous decoding of $(\mathbf{x}_{12}, \mathbf{x}_3)$ is performed while treating other signals as noise.

And then, \mathbf{x}_{11} and \mathbf{x}_{10} are decoded successively. For reliable decoding, code rates should satisfy

$$\begin{aligned} R_{12} &\leq I_1 = \frac{1}{2} \log \left(1 + \frac{P - N_2 - N_3}{N_1 + N_2 + 2N_3} \right) \\ R_{21} &\leq I_2 = \frac{1}{2} \log \left(1 + \frac{P - N_3}{N_1 + N_2 + 2N_3} \right) \\ R_{12} + R_{21} &\leq I_3 = \frac{1}{2} \log \left(1 + \frac{2P - N_2 - 2N_3}{N_1 + N_2 + 2N_3} \right) \\ R_{11} &\leq I_4 = \frac{1}{2} \log \left(1 + \frac{N_3}{N_1 + N_2} \right) \\ R_{20} &\leq I_5 = \frac{1}{2} \log \left(1 + \frac{N_3}{N_1 + N_2} \right) \\ R_{11} + R_{20} &\leq I_6 = \frac{1}{2} \log \left(1 + \frac{2N_3}{N_1 + N_2} \right) \\ R_{10} &\leq I_7 = \frac{1}{2} \log \left(1 + \frac{N_2}{N_1} \right) \end{aligned}$$

at receiver 1,

$$\begin{aligned} R_{21} &\leq I_8 = \frac{1}{2} \log \left(1 + \frac{P - N_3}{N_2 + N_3} \right) \\ R_3 &\leq I_9 = \frac{1}{2} \log \left(1 + \frac{P}{N_2 + N_3} \right) \\ R_{21} + R_3 &\leq I_{10} = \frac{1}{2} \log \left(1 + \frac{2P - N_3}{N_2 + N_3} \right) \\ R_{20} &\leq I_{11} = \frac{1}{2} \log \left(1 + \frac{N_3}{N_2} \right) \end{aligned}$$

at receiver 2,

$$\begin{aligned} R_{12} &\leq I_{12} = \frac{1}{2} \log \left(1 + \frac{P - N_2 - N_3}{N_2 + 2N_3} \right) \\ R_3 &\leq I_{13} = \frac{1}{2} \log \left(1 + \frac{P}{N_2 + 2N_3} \right) \\ R_{12} + R_3 &\leq I_{14} = \frac{1}{2} \log \left(1 + \frac{2P - N_2 - N_3}{N_2 + 2N_3} \right) \end{aligned}$$

at receiver 3. Putting together,

$$\begin{aligned} R_{12} &\leq T_1 = \min\{I_1, I_{12}\} = I_1 \\ R_{21} &\leq T_2 = \min\{I_2, I_8\} = I_2 \\ R_3 &\leq T_3 = \min\{I_9, I_{13}\} = I_{13} \\ R_{12} + R_{21} &\leq T_4 = I_3 \\ R_{12} + R_3 &\leq T_5 = I_{14} \\ R_{21} + R_3 &\leq T_6 = I_{10} \end{aligned}$$

at the top layer,

$$\begin{aligned} R_{11} &\leq T_7 = I_4 \\ R_{20} &\leq T_8 = \min\{I_5, I_{11}\} = I_5 \\ R_{11} + R_{20} &\leq T_9 = I_6 \end{aligned}$$

at the mid-layer,

$$R_{10} \leq T_{10} = I_7$$

at the bottom layer. Note that the rate variables are not coupled between layers. We get the achievable rate region

$$\begin{aligned}
R_1 &= R_{12} + R_{11} + R_{10} \leq T_1 + T_7 + T_{10} \\
R_2 &= R_{21} + R_{20} \leq T_2 + T_8 \\
R_3 &\leq T_3 \\
R_1 + R_2 &\leq T_4 + T_9 + T_{10} \\
R_1 + R_3 &\leq T_5 + T_7 + T_{10} \\
R_2 + R_3 &\leq T_6 + T_8.
\end{aligned}$$

This region includes the following region.

$$\begin{aligned}
R_1 &\leq \frac{1}{2} \log \left(2 + \frac{P}{N_1} \right) - \frac{1}{2} \\
R_2 &\leq \frac{1}{2} \log \left(2 + \frac{P}{N_2} \right) - 1 \\
R_3 &\leq \frac{1}{2} \log \left(3 + \frac{P}{N_3} \right) - \frac{1}{2} \log(3) \\
R_1 + R_2 &\leq \frac{1}{2} \log \left(1 + \frac{2P}{N_1} \right) \\
R_1 + R_3 &\leq \frac{1}{2} \log \left(1 + \frac{2P}{N_1} \right) - \frac{1}{2} \\
R_2 + R_3 &\leq \frac{1}{2} \log \left(1 + \frac{2P}{N_2} \right) - \frac{1}{2}.
\end{aligned}$$

Therefore, we can conclude the capacity region to within one bit.

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